

CENTRAL REFLECTIONS AND NILPOTENCY IN EXACT MAL'CEV CATEGORIES

CLEMENS BERGER AND DOMINIQUE BOURN

ABSTRACT. A general notion of nilpotency is studied in the context of exact Mal'cev categories, subsuming the notions of nilpotent group and of nilpotent Lie algebra as special cases. The nilpotent objects of class $\leq n$ are shown to form a Birkhoff subcategory enjoying several most specific properties.

The relationship between nilpotency and identity functors with bounded degree is investigated, using an algebraic version of Goodwillie's cubical cross-effects and a new concept of n -folded object.

INTRODUCTION

This text investigates *nilpotency* in the framework of *exact Mal'cev categories*. Our purpose is twofold: basic phenomena of nilpotency are treated through universal properties rather than through commutator calculus, emphasizing the fundamental role played by *central extensions*; nilpotency is then linked to Goodwillie's cubical cross-effects [29], leading to a global understanding of nilpotency in terms of identity functors with bounded degree.

Abstract features of nilpotency, based on a commutator calculus of congruences, have been studied by Smith [59] in the context of Mal'cev varieties, and by Freese-McKenzie [27] in the broader context of congruence modular varieties. Quite a bit earlier, Higgins [41] and Huq [43] had developed a commutator calculus for subobjects, which in many respects deserved a similar purpose, see Mantovani-Metere [54] and Hartl, Loiseau, Van der Linden [38, 39] for a conceptual modern treatment. It took a while to conciliate the “congruence commutator” and “subobject commutator” approaches. An important step was the discovery by Carboni, Kelly, Lambek and Pedicchio [18, 20] that Mal'cev's permutability condition for congruences not only suits for varieties, where it amounts to the existence of a Mal'cev term [53], but more generally for any regular category, since it permits diagram chasing and implies in particular the validity of the extremely useful 3×3 -lemma, cf. [8, 50].

The Mal'cev condition itself has been characterized by the second author [6, 7] via the *fibration of points* and shown to be a consequence of *protomodularity*. In a pointed setting, protomodularity amounts to the condition that section and kernel-inclusion of a split epimorphism form a strongly epimorphic cospan. Janelidze-Márki-Tholen [47] introduced the term *semi-abelian category* for an exact protomodular category which is pointed and has binary sums; they show that semi-abelian categories can be characterized by axioms on kernels and cokernels which are reminiscent of those used in the beginning of the theory of abelian categories.

Date: January 19, 2016.

1991 Mathematics Subject Classification. 17B30, 18D35, 18E10, 18G50, 20F18.

Key words and phrases. Nilpotency, Mal'cev category, central extension, cubical cross-effect.

Examples of semi-abelian categories include the categories of groups, of loops, of augmented algebras, of Lie algebras, and of cocommutative Hopf algebras over a field of characteristic zero [31]. Since *pointed categories with binary sums* will play an important role throughout we shall call them σ -pointed for short.

Although our strongest results are formulated for semi-abelian categories we work as long as possible with exact Mal'cev categories, privileging kernel relations and their quotients over kernels and short exact sequences. A central extension is defined as a regular epimorphism with central kernel relation, and n -nilpotent objects are those which can be obtained as an n -fold central extension of the terminal object. This yields the usual notions of n -nilpotent group, n -nilpotent Lie algebra and n -nilpotent loop [16]. We call a category n -nilpotent if all its objects are n -nilpotent.

As a first general result we obtain that for an exact Mal'cev category with binary sums, the full subcategory spanned by the n -nilpotent objects is a *reflective Birkhoff subcategory* (cf. Theorem 2.14). Throughout, we denote the reflection into the subcategory of n -nilpotent objects by I^n and the unit of the adjunction at an object X by $\eta_X^n : X \rightarrow I^n(X)$. Since an n -nilpotent object is a fortiori $(n+1)$ -nilpotent, the different reflections assemble into the following *nilpotency tower*

$$\begin{array}{c} X \\ \swarrow \quad \searrow \quad \downarrow \quad \swarrow \quad \searrow \\ \star \leftarrow I^1(X) \leftarrow I^2(X) \leftarrow \dots \leftarrow I^n(X) \leftarrow I^{n+1}(X) \leftarrow \dots \end{array}$$

$\eta_X^1 \quad \eta_X^2 \quad \eta_X^n \quad \eta_X^{n+1}$

in which the successive quotient maps $I^{n+1}(X) \rightarrow I^n(X)$ are central extensions.

Among σ -pointed exact Mal'cev categories we characterize the n -nilpotent one's as those for which the canonical comparison maps $\theta_{X,Y} : X + Y \rightarrow X \times Y$ are $(n-1)$ -fold central extensions (cf. Theorem 4.3). The *nilpotency class* of a σ -pointed exact Mal'cev category measures thus the discrepancy between binary sum and binary product. If $n = 1$, binary sum and binary product coincide, and all objects are abelian group objects. A σ -pointed exact Mal'cev category is 1-nilpotent if and only if it is an abelian category (cf. Corollary 4.4). The unit $\eta_X^1 : X \rightarrow I^1(X)$ of the first Birkhoff reflection is thus *abelianization* (cf. Proposition 4.2).

For a 2-nilpotent exact Mal'cev category, the unit $\eta_X^1 : X \rightarrow I^1(X)$ is pointwise a central extension. We call such reflections *central*. It is important that central reflections of exact Mal'cev categories actually have a stronger property: their unit is pointwise *affine* (cf. Theorem 3.5). A morphism f is affine if base-change along f with respect to the fibration of points [6, 7] is an equivalence of categories. For instance, in an additive category with pullbacks all morphisms are affine. In a semi-abelian category, affine extensions $f : X \rightarrow Y$ have the characteristic property that $f \diamond Z : X \diamond Z \rightarrow Y \diamond Z$ is invertible for all objects Z (cf. Proposition 3.13) where $X \diamond Z$ denotes the kernel of $\theta_{X,Z}$, the so-called *co-smash product*, cf. [17]. All morphisms inverted by a central reflection are affine. The comparison maps $\theta_{X,Y} : X + Y \rightarrow X \times Y$ in a 2-nilpotent exact Mal'cev category are thus affine.

Goodwillie [29] defined for endofunctors of the category of based spaces a tower of approximating endofunctors such that the successive fibres of this tower take values in infinite loop spaces. Infinite loop spaces play a similar role in the homotopy theory of based spaces as abelian group objects in pointed Mal'cev categories. The Goodwillie tower of the identity functor of the category of based spaces is thus a

homotopical analogue of the nilpotency tower of a σ -pointed exact Mal'cev category. One of the aims of our study of nilpotency was to get a deeper understanding of this analogy. The n -th approximating endofunctor of the Goodwillie tower is universally n -excisive, and n -excisive functors are *almost* characterized by the property of having vanishing *cross-effects* of order $n + 1$ or, as we shall say, by the property of being of degree $\leq n$. Although n -excisive functors do not have an algebraic analogue, functors of degree $\leq n$ do. In particular, Goodwillie's cubical definition of cross-effects translates well into our algebraic setting.

For each $(n + 1)$ -tupel (X_1, \dots, X_{n+1}) of objects of a σ -pointed category and each based endofunctor F we define an $(n + 1)$ -dimensional cube $\Xi_{X_1, \dots, X_{n+1}}^F$ consisting of the images $F(X_{i_1} + \dots + X_{i_k})$ for all subsequences of (X_1, \dots, X_{n+1}) together with the obvious contraction maps. We say that the functor F is of degree $\leq n$ if these $(n + 1)$ -dimensional cubes are limit-cubes for all choices of (X_1, \dots, X_{n+1}) . A based functor F is thus of degree ≤ 1 if and only if $F(X_1 + X_2) \cong F(X_1) \times F(X_2)$ for all X_1, X_2 which is the usual definition of a *linear* functor.

If pullbacks exist, the $(n + 1)$ -st *cross-effect* $cr_{n+1}^F(X_1, \dots, X_{n+1})$ is defined as the *total kernel* of the $(n + 1)$ -cube $\Xi_{X_1, \dots, X_{n+1}}^F$, i.e. as the kernel of the canonical comparison map $\theta_{X_1, \dots, X_{n+1}}^F : F(X_1 + \dots + X_{n+1}) \rightarrow P_{X_1, \dots, X_{n+1}}^F$ to the limit of the punctured $(n + 1)$ -cube. In particular, the second cross-effect $cr_2(X, Y)$ of the identity coincides with the co-smash product $X \diamond Y$, cf. Carboni-Janelidze [17].

An endofunctor of a semi-abelian (or homological [5]) category is of degree $\leq n$ if and only if all its cross-effects of order $n + 1$ vanish. For functors taking values in abelian categories, our cross-effects coincide with the original one's of Eilenberg-Mac Lane [24] (cf. Remark 6.2). For functors taking values in σ -pointed categories with pullbacks, our cross-effects coincide with those of Hartl-Loiseau [38] and Hartl-Van der Linden [39] defined as kernel intersections (cf. Definition 5.1).

A Goodwillie type characterization of the nilpotency tower amounts to the property that for each n , the reflection I^n into the Birkhoff subcategory of n -nilpotent objects is the universal endofunctor of degree $\leq n$. In fact, any endofunctor of degree $\leq n$ of a σ -pointed exact Mal'cev category takes values in n -nilpotent objects (cf. Proposition 6.5). The reflection I^n is of degree $\leq n$ if and only if the identity functor of the Birkhoff subcategory of n -nilpotent objects itself is of degree $\leq n$. It is this last property we have mainly been investigating in the present article.

The property holds for $n = 1$ because the identity functor of an abelian category is linear. There are however 2-nilpotent semi-abelian categories which are not *quadratic* (i.e. do not have an identity functor of degree ≤ 2), as for instance the semi-abelian category of 2-nilpotent *Moufang loops* (cf. Section 6.28). We obtain a precise criterion for when a σ -pointed exact Mal'cev category is quadratic: this is the case if and only if the category is 2-nilpotent *and* algebraically distributive, i.e. endowed with isomorphisms $(X \times Z) +_Z (Y \times Z) \cong (X + Y) \times Z$ for all objects X, Y, Z (cf. Corollary 5.19). Since algebraic distributivity is preserved under Birkhoff reflection, the subcategory of 2-nilpotent objects of an algebraically distributive semi-abelian category is always quadratic (cf. Theorem 5.21).

Algebraic distributivity is a consequence of the *existence of centralizers for subobjects* as shown by Gray and the second author [14]. For pointed Mal'cev categories, it also follows from *algebraic coherence* in the sense of Cigoli-Gray-Van der Linden [21]. Our quadraticity result implies that iterated Huq commutator $[X, [X, X]]$ and

ternary Higgins commutator $[X, X, X]$ coincide for each object X of an algebraically distributive semi-abelian category (cf. Corollary 5.22 and [21, Corollary 7.2]).

There is a remarkable duality for σ -pointed 2-nilpotent exact Mal'cev categories: algebraic distributivity amounts to algebraic codistributivity, i.e. to isomorphisms $(X \times Y) + Z \cong (X + Z) \times_Z (Y + Z)$ for all X, Y, Z (cf. Proposition 5.18). Indeed, the difference between 2-nilpotency and quadraticity is precisely algebraic codistributivity (cf. Theorem 5.6). The extension of this duality to all $n \geq 2$ is the main step in relating general nilpotency to identity functors with bounded degree.

The following characterization is very useful: The identity functor of a σ -pointed exact Mal'cev category \mathbb{E} is of degree $\leq n$ if and only if all its objects are n -folded (cf. Proposition 6.5). An object is n -folded (cf. Definition 6.3) if the $(n+1)$ -st *folding map* $\delta_{n+1}^X : X + \cdots + X \rightarrow X$ factors through the comparison map $\theta_{X, \dots, X} : X + \cdots + X \rightarrow P_{X, \dots, X}$. In a varietal context this can be expressed in combinatorial terms (cf. Remark 6.4). The full subcategory $\text{Fld}^n(\mathbb{E})$ spanned by the n -folded objects is a reflective Birkhoff subcategory of \mathbb{E} , and the reflection $J^n : \mathbb{E} \rightarrow \text{Fld}^n(\mathbb{E})$ is the universal endofunctor of degree $\leq n$ (cf. Theorem 6.8). Every n -folded object is n -nilpotent (cf. Proposition 6.14) while the converse holds if and only if the other Birkhoff reflection $I^n : \mathbb{E} \rightarrow \text{Nil}^n(\mathbb{E})$ is also of degree $\leq n$.

The nilpotency tower fulfills thus a Goodwillie type universal property if and only if n -nilpotency amounts to n -foldedness for all n . In order to show that semi-abelianity is not enough in general, we exhibit a Moufang loop of order 16 (subloop of Cayley's octonions) which is 2-nilpotent but not 2-folded (cf. Section 6.28).

We did not find a simple categorical structure that would entail the equivalence between n -nilpotency and n -foldedness for all n . As a first step in this direction we show that an n -nilpotent semi-abelian category has an identity of degree $\leq n$ if and only if its n -th cross-effect is *multilinear* (cf. Theorem 6.24). We also show that the nilpotency tower has the desired universal property if and only if it is *homogeneous*, i.e. for each n , the n -th kernel functor is of degree $\leq n$ (cf. Theorem 6.26). This is preserved under Birkhoff reflection (cf. Theorem 6.27). The categories of groups and of Lie algebras have homogeneous nilpotency towers so that a group, resp. Lie algebra is n -nilpotent if and only if it is n -folded, and the Birkhoff reflection I^n is here indeed the universal endofunctor of degree $\leq n$. The category of *triality groups* [23, 28, 37] also has a homogeneous nilpotency tower although it contains the category of Moufang loops as a full coreflective subcategory (cf. Section 6.28).

There are several pursuing ideas closely related to the contents of this article which we hope to address in future work. Let us comment on two of them:

The associated graded object of the nilpotency tower $\oplus_{n \geq 1} K[I^n(X) \rightarrow I^{n-1}(X)]$ is a functor in X taking values in graded abelian group objects. For the category of groups this functor actually takes values in graded Lie rings and as such preserves n -nilpotent objects and free objects, cf. Lazard [51]. It is likely that for a large class of semi-abelian categories, the associated graded object of the nilpotency tower carries a similar algebraic structure. It would be interesting to establish the relationship between this algebraic structure and the cross-effects of the identity functor.

Combining [18, Theorem 4.2] of Carboni-Kelly-Pedicchio with [58, Theorem IV.4] of Quillen shows that the simplicial objects of a pointed Mal'cev variety carry a Quillen model structure for which the weak equivalences are the simplicial maps inducing a quasi-isomorphism on Moore complexes. In this model structure regular epimorphisms are fibrations, and the trivial fibrations are precisely the regular

epimorphisms for which the kernel is homotopically trivial. This implies that Goodwillie's homotopical cross-effects coincide here with our algebraic cross-effects.

Several notions of *homotopical nilpotency* (cf. [2]) are now available. The first is the least integer n for which the unit $\eta_{X_\bullet}^n : X_\bullet \rightarrow I^n(X_\bullet)$ is a trivial fibration, the second (resp. third) is the least integer n for which X_\bullet is homotopically n -folded (resp. the value of an n -excisive approximation of the identity). The first is a lower bound for the second, and the second is a lower bound for the third invariant. For *simplicial groups* the first invariant recovers the Berstein-Ganea nilpotency for loop spaces [3], the second the cocategory of Hovey [42], and the third the Biedermann-Dwyer nilpotency for homotopy nilpotent groups [4]. Similar chains of inequalities have recently been studied by Eldred [25] and Costoya-Scherer-Viruel [22].

Acknowledgements. We are grateful to Georg Biedermann, Rosona Eldred, Marino Gran, James Gray, Jonathan Hall, George Janelidze, Daniel Tanré and Tim Van der Linden for helpful discussions. Special thanks are due to Manfred Hartl whose seminar talk in September 2013 in Nice was the starting point for this work. The first author gratefully acknowledges financial support of the French ANR grant HOGT.

1. CENTRAL EXTENSIONS AND REGULAR PUSHOUTS

In this introductory section we review the notion of *central equivalence relation* and study basic properties of the associated class of *central extensions*, needed for our treatment of nilpotency. By central extension we mean a regular epimorphism with central kernel relation [9, 12, 13]. This algebraic concept of central extension has to be distinguished from the axiomatic concept of Janelidze-Kelly [44] which is based on a previously chosen admissible Birkhoff subcategory. Nevertheless, it is known that with respect to the Birkhoff subcategory of *abelian group objects*, the two approaches yield the same class of central extensions in any congruence modular variety (cf. [45, 46]) as well as in any semi-abelian category (cf. [12]).

We assume throughout that our ambient category is a *Mal'cev category*, i.e. a finitely complete category in which every reflexive relation is an equivalence relation, cf. [5, 7, 18]. Most of the material of this section is well-known to the expert, and treated in some detail here mainly to fix notation and terminology.

One exception is Section 1.22 which establishes an “algebraic” *Beck-Chevalley condition* for pushouts of regular epimorphisms in exact Mal'cev categories, dual to the familiar Beck-Chevalley condition for pullbacks of monomorphisms in elementary toposes. In recent and independent work, Gran-Rodelo [32] consider a weaker form of this condition and show that it characterizes regular Goursat categories.

1.1. Smith commutator of equivalence relations. –

An *equivalence relation* R on X will be denoted as a reflexive graph $(p_0, p_1) : R \rightrightarrows X$ with section $s_0 : X \rightarrow R$, but whenever convenient we shall consider R as a subobject of $X \times X$. By a *cartesian map* of equivalence relations $(X, R) \rightarrow (Y, S)$ we mean a cartesian natural transformation of the underlying reflexive graphs.

A particularly important equivalence relation is the *kernel relation* $R[f]$ of a morphism $f : X \rightarrow Y$ which is part of the following diagram:

$$R[f] \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_0} \end{array} X \xrightarrow{f} Y.$$

The *discrete* equivalence relation Δ_X on X is the kernel relation $R[1_X]$ of the identity map $1_X : X \rightarrow X$. The *indiscrete* equivalence relation ∇_X on X is the kernel relation $R[\omega_X]$ of the unique map ω_X from X to a terminal object.

Two equivalence relations R, S on the same object X are said to *centralize each other* if the square

$$\begin{array}{ccc} R \times_X S & \xleftarrow{(s_0^R, 1_S)} & S \\ \uparrow (1_R, s_0^S) & \searrow p & \downarrow p_1^S \\ R & \xrightarrow{p_0^R} & X \end{array}$$

admits a (necessarily unique) filler which makes the diagram commute, cf. [13, 56].

In set-theoretical terms such a filler amounts to the existence of a “*partial Mal’cev operation*” on X , namely (considering $R \times_X S$ as a subobject of $X \times X \times X$) a ternary operation $p : R \times_X S \rightarrow X$ such that $x = p(x, y, y)$ and $p(x, x, y) = y$. We shall follow Marino Gran and the second author in calling $p : R \times_X S \rightarrow X$ a *connector* between R and S , cf. [12, 13].

In a finitely cocomplete regular Mal’cev category, there exists for each pair (R, S) of equivalence relations on X a finest effective equivalence relation $[R, S]$ on X such that R and S centralize each other in the quotient $X/[R, S]$. This universal equivalence relation is the so-called *Smith commutator* of R and S , cf. [9, 13, 56, 59].

In these terms R and S centralize each other precisely when $[R, S] = \Delta_X$. The Smith commutator is monotone in each variable and satisfies

$$[R, S] = [S, R] \text{ and } f([R, S]) \subset [f(R), f(S)]$$

for each regular epimorphism $f : X \rightarrow Y$, where $f(R)$ denotes the direct image of the subobject $R \subset X \times X$ under the regular epimorphism $f \times f : X \times X \rightarrow Y \times Y$. The Mal’cev condition implies that this direct image represents an equivalence relation on Y . Note that equality $f([R, S]) = [f(R), f(S)]$ holds if and only if the direct image $f([R, S])$ is an *effective* equivalence relation on Y .

1.2. Central equivalence relations and central extensions. An equivalence relation R on X is said to be *central* if $[R, \nabla_X] = \Delta_X$. A *central extension* is by definition a regular epimorphism with central kernel relation. An *n-fold central extension* is the composite of n central extensions. An *n-fold centrally decomposable morphism* is the composite of n morphisms with central kernel relation.

The indiscrete equivalence relation ∇_X is a central equivalence relation precisely when X admits an *internal Mal’cev operation* $p : X \times X \times X \rightarrow X$. In pointed Mal’cev categories such a Mal’cev operation amounts to an *abelian group* structure on X , cf. [5, Proposition 2.3.8]. An object X of a pointed Mal’cev category $(\mathbb{D}, \star_{\mathbb{D}})$ is thus an abelian group object if and only if the map $X \rightarrow \star_{\mathbb{D}}$ is a central extension.

Central equivalence relations are closed under binary products and inverse image along monomorphisms. In a regular Mal’cev category, central equivalence relations are closed under direct image, cf. [13, Proposition 4.2] and [5, Proposition 2.6.15].

Lemma 1.3. *In a regular Mal’cev category, an n-fold centrally decomposable morphism can be written as an n-fold central extension followed by a monomorphism.*

Proof. It suffices to show that a monomorphism ψ followed by a central extension ϕ can be rewritten as a central extension ϕ' followed by a monomorphism ψ' . Indeed, the kernel relation $R[\phi\psi]$ is central, being the restriction $\psi^{-1}(R[\phi])$ of the central

equivalence relation $R[\phi]$ along the monomorphism ψ ; therefore, by regularity, one obtains $\phi\psi = \psi'\phi'$ where ϕ' is quotienting by the kernel relation $R[\phi\psi]$. \square

Lemma 1.4. *In a Mal'cev category, morphisms with central kernel relation are closed under pullback. In a regular Mal'cev category, central extensions are closed under pullback.*

Proof. It suffices to show the first statement. In the following diagram,

$$\begin{array}{ccccc} R[f'] & \xrightleftharpoons[p'_0]{p'_1} & X' & \xrightarrow{f'} & Y' \\ R(x,y) \downarrow & & \downarrow x & & \downarrow y \\ R[f] & \xrightleftharpoons[p_0]{p_1} & X & \xrightarrow{f} & Y \end{array}$$

if the right square is a pullback, then the left square is a cartesian natural transformation of reflexive graphs. This permits to lift the connector $p : R[f] \times_X \nabla_X \rightarrow X$ so as to obtain a connector $p' : R[f'] \times_{X'} \nabla_{X'} \rightarrow X'$. \square

Lemma 1.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in a Mal'cev category.*

If gf is a morphism with central kernel relation then so is f . More generally, if gf is n -fold centrally decomposable then so is f .

Proof. Since $R[f] \subset R[gf]$, the commutation relation $[R[gf], \nabla_X] = \Delta_X$ implies the commutation relation $[R[f], \nabla_X] = \Delta_X$. Assume now $gf = k_n \cdots k_1$ where each k_i is a morphism with central kernel relation. In the following pullback

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & X \\ \psi \downarrow & \phi \dashrightarrow & \downarrow gf \\ Y & \xrightarrow{g} & Z \end{array}$$

ϕ is the unique map such that $\gamma\phi = 1_X$ and $\psi\phi = f$. If we denote by h_i the morphism with central kernel relation obtained by pulling back k_i along g , we get $f = h_n \cdots h_2(h_1\phi)$. Since ϕ is a monomorphism, the kernel relation $R[h_1\phi] = \phi^{-1}(R[h_1])$ is central, and hence f is the composite of n morphisms with central kernel relation. \square

Proposition 1.6 (Corollary 3.3 in [9]). *In a finitely cocomplete regular Mal'cev category, each morphism $f : X \rightarrow Y$ factors canonically as in*

$$\begin{array}{ccc} X & \xrightarrow{\eta_f} & X/[\nabla_X, R[f]] \\ f \downarrow & \searrow \zeta_f & \downarrow \\ Y & \xleftarrow{\quad} & X/R[f] \end{array}$$

where η_f is a regular epimorphism and ζ_f has a central kernel relation. If f is a regular epimorphism then ζ_f is a central extension.

This factorization has the following universal property. Any commutative diagram of undotted arrows as below (with $Z_f = X/[\nabla_X, R[f]]$), such that ζ' has a

central kernel relation, produces a unique dotted map

$$\begin{array}{ccccc}
 X & \xrightarrow{x} & X' & & \\
 \eta_f \searrow & & \eta' \searrow & & \\
 & Z_f & \cdots \cdots z \cdots \cdots & Z' & \\
 \zeta_f \swarrow & & \zeta' \swarrow & & \\
 Y & \xrightarrow{y} & Y' & &
 \end{array}$$

rendering the whole diagram commutative.

Proposition 1.7. *In a finitely cocomplete regular Mal'cev category, each morphism has a universal factorisation into a regular epimorphism followed by an n -fold centrally decomposable morphism. Each n -fold central extension has an initial factorization into n central extensions.*

Proof. We proceed by induction, the case $n = 1$ being treated in Proposition 1.6. Suppose the assertion holds up to level $n - 1$ with $f = \zeta_f^{n-1} \zeta_f^{n-2} \cdots \zeta_f^1 \eta_f^1$ and η_f^1 a regular epimorphism. Take the universal factorization of η_f^1 . The universality of this new factorization is then a straightforward consequence of the induction hypothesis and the universal property stated in Proposition 1.6.

Starting with an n -fold central extension f , its universal factorization through a composite of $n - 1$ morphisms with central kernel relation makes the regular epimorphism η_f^1 a central extension by Lemma 1.5, and therefore produces a factorization into n central extensions which is easily seen to be the initial one. \square

1.8. Regular pushouts. In a regular category, any pullback square of regular epimorphisms is also a pushout square, as follows from the pullback stability of regular epimorphisms. In particular, a commuting square of regular epimorphisms

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 x \downarrow & & \downarrow y \\
 X' & \xrightarrow{f'} & Y'
 \end{array}$$

is a pushout whenever the comparison map $(x, f) : X \rightarrow X' \times_{Y'} Y$ to the pullback of y along f' is a regular epimorphism. Such pushouts will be called *regular*, cf. [8]. A regular pushout induces in particular regular epimorphisms on kernel relations which we shall denote $R(x, y) : R[f] \rightarrow R[f']$ and $R(f, f') : R[x] \rightarrow R[y]$.

For a regular Mal'cev category the following more precise result holds.

Proposition 1.9 (cf. Proposition 3.3 in [8]). *In a regular Mal'cev category, a commuting square of regular epimorphisms like in (1.8) is a regular pushout if and only if one of the following three equivalent conditions holds:*

- (a) *the comparison map $X \rightarrow X' \times_{Y'} Y$ is a regular epimorphism;*
- (b) *the induced map $R(x, y) : R[f] \rightarrow R[f']$ is a regular epimorphism;*
- (c) *the induced map $R(f, f') : R[x] \rightarrow R[y]$ is a regular epimorphism.*

Accordingly, central extensions are closed under regular pushout.

Proof. We already mentioned that (a) implies (b) and (c) in any regular category. It remains to be shown that in a regular Mal'cev category (b) or (c) implies (a).

For this, it is useful to notice that in a regular category condition (a) holds if and only if the composite relation $R[f] \circ R[x]$ equals the kernel relation of the diagonal of the square by Theorem 5.2 of Carboni-Kelly-Pedicchio [18]. Since this kernel relation is given by $x^{-1}(R[f'])$, and condition (b) just means that $x(R[f]) = R[f']$, it suffices to establish the identity $R[f] \circ R[x] = x^{-1}(x(R[f]))$. In a regular Mal'cev category, the composition of equivalence relations is symmetric and coincides with their *join*. The join $R[f] \vee R[x]$ is easily identified with $x^{-1}(x(R[f]))$.

The second assertion follows from (b) resp. (c) and the closure of central kernel relations under direct image in regular Mal'cev categories. \square

Corollary 1.10. *In a regular Mal'cev category, any commutative square*

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{s} \end{array} & Y \\ x \downarrow & & \downarrow y \\ X' & \begin{array}{c} \xrightarrow{f'} \\ \xleftarrow{s'} \end{array} & Y' \end{array}$$

with a parallel pair of regular epimorphisms and a parallel pair of split epimorphisms is a regular pushout.

Proof. The induced map $R(f, f') : R[x] \rightarrow R[y]$ is a split and hence regular epimorphism so that the pushout is regular by Proposition 1.9. \square

Corollary 1.11. *In an exact Mal'cev category, pushouts of regular epimorphisms along regular epimorphisms exist and are regular pushouts.*

Proof. Given a pair (f, x) of regular epimorphisms with common domain, consider the following diagram

$$\begin{array}{ccccc} R[f] & \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_0} \end{array} & X & \xrightarrow{f} & Y \\ \bar{x} \downarrow & & \downarrow x & & \downarrow y \\ S & \begin{array}{c} \xrightarrow{q_1} \\ \xleftarrow{q_0} \end{array} & X' & \xrightarrow{f'} & Y' \end{array}$$

in which S denotes the direct image $x(R[f])$. By exactness, this equivalence relation on X' has a quotient Y' . The induced right square is then a regular pushout. \square

Remark 1.12. It follows from [7] that Corollary 1.10 characterizes regular Mal'cev categories among regular categories, while [18, Theorem 5.7] shows that Corollary 1.11 characterizes exact Mal'cev categories among regular categories.

Remark 1.13. It is worthwhile noting that in *any* category a commuting square of epimorphisms in which one parallel pair admits compatible sections is automatically a pushout square. Dually, a commuting square of monomorphisms in which one parallel pair admits compatible retractions is automatically a pullback square.

Lemma 1.14. *In a pointed regular category, a regular pushout induces a regular epimorphism on parallel kernels*

$$\begin{array}{ccccc} K[f] & \xrightarrow{\quad} & X & \xrightarrow{f} & Y \\ K(x, y) \downarrow & & \downarrow x & & \downarrow y \\ K[f'] & \xrightarrow{\quad} & X' & \xrightarrow{f'} & Y' \end{array}$$

so that the kernel $K[f']$ of f' is the image under x of the kernel $K[f]$ of f .

Proof. This follows from the fact that, in any pointed category, the induced map on kernels factors as a pullback of the comparison map $(x, f) : X \twoheadrightarrow X' \times_{Y'} Y$ followed by an isomorphism. \square

Proposition 1.15. *In a finitely cocomplete regular Mal'cev category, consider the following diagram of pushouts*

$$\begin{array}{ccccc} X & \xrightarrow{\eta_f} & Z_f & \xrightarrow{\zeta_f} & Y \\ x \downarrow & & \downarrow z & & \downarrow y \\ X' & \xrightarrow{\eta_{f'}} & Z_{f'} & \xrightarrow{\zeta_{f'}} & Y' \end{array}$$

in which the upper row represents the universal factorisation of the regular epimorphism f into a regular epimorphism η_f followed by a central extension ζ_f .

If the outer rectangle is a regular pushout then the right square as well, and the lower row represents the universal factorisation 1.6 of its composite $f' = \zeta_{f'}\eta_{f'}$.

Proof. Since on vertical kernel relations $R[x] \rightarrow R[z] \rightarrow R[y]$ we get a regular epimorphism, the second one $R[z] \rightarrow R[y]$ is a regular epimorphism as well, and the right square is a regular pushout by Proposition 1.9. Therefore, $\zeta_{f'}$ is a central extension by Corollary 1.10. It remains to be shown that the lower row fulfills the universal property of factorization 1.6. Consider the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_f} & Z_f & \xrightarrow{\zeta_f} & Y \\ x \downarrow & & \downarrow z & & \downarrow y \\ X' & \xrightarrow{\eta_{f'}} & Z_{f'} & \xrightarrow{\zeta_{f'}} & Y' \\ & \searrow \eta & \downarrow z'' & \nearrow \zeta & \\ & & Z' & & \end{array}$$

with central extension ζ . According to Proposition 1.6, there is a unique dotted factorization z'' making the diagram commute. Since the left square is a pushout, z'' factors uniquely and consistently through $z' : Z_{f'} \rightarrow Z'$, showing that the lower row has indeed the required universal property. \square

Proposition 1.16. *In a finitely cocomplete exact Mal'cev category, the universal factorization of a regular epimorphism through an n -fold central extension is preserved under pushout along regular epimorphisms.*

Proof. Let us consider the following diagram of pushouts

$$\begin{array}{ccccccc} X & \xrightarrow{\eta} & Z_n & \xrightarrow{\zeta_n} & Z_{n-1} & \dots & Z_1 \xrightarrow{\zeta_1} Y \\ x \downarrow & & \downarrow z_n & & \downarrow z_n & & \downarrow z_1 \downarrow y \\ X' & \xrightarrow{\eta'} & Z'_n & \xrightarrow{\zeta'_n} & Z'_{n-1} & \dots & Z'_1 \xrightarrow{\zeta'_1} Y' \end{array}$$

in which the upper row is the universal factorization 1.7 of a regular epimorphism $f : X \rightarrow Y$ through an n -fold central extension. By Corollary 1.11, all pushouts are regular. Therefore, the morphisms $\zeta'_k : Z'_k \rightarrow Z'_{k-1}$ are central extensions for all k . It remains to be shown that the lower row satisfies the universal property of the

factorisation 1.7 of $f' : X' \rightarrow Y'$ through an n -fold central extension. This follows induction on n beginning with the case $n = 1$ proved in Proposition 1.15. \square

Proposition 1.17. *Let \mathbb{D} be an exact Mal'cev category. Consider the following diagram of pushouts*

$$\begin{array}{ccccccc} X & \xrightarrow{f_n} & X_{n-1} & \xrightarrow{f_{n-1}} & X_{n-2} & \cdots & X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} Y \\ x_n \downarrow & & x_{n-1} \downarrow & & x_{n-2} \downarrow & & \downarrow x_1 \quad \downarrow x_0 \quad \downarrow y \\ X' & \xrightarrow{f'_n} & X'_{n-1} & \xrightarrow{f'_{n-1}} & X'_{n-2} & \cdots & X'_1 \xrightarrow{f'_1} X'_0 \xrightarrow{f'_0} Y' \end{array}$$

in which $x_n : X \rightarrow X'$ is a regular epimorphism.

If the upper row represents an n -fold central extension of the regular epimorphism $f_0 : X_0 \rightarrow Y$ in the slice category \mathbb{D}/Y then the lower row represents an n -fold central extension of $f'_0 : X'_0 \rightarrow Y'$ in the slice category \mathbb{D}/Y' .

Proof. Let us set $\phi_i = f_0 \cdot f_1 \cdots f_i$ and $\phi'_i = f'_0 \cdot f'_1 \cdots f'_i$. Since the indiscrete equivalence relation ∇_{f_0} on the object $f_0 : X_0 \rightarrow Y$ of the slice category \mathbb{D}/Y is given by $R[f_0]$, our assumption on the upper row translates into the conditions

$$[R[f_i], R[\phi_i]] = \Delta_{X_i} \text{ for } 1 \leq i \leq n.$$

Since any of the rectangles is a regular pushout by Corollary 1.11, we get $x_i(R[f_i]) = R[f'_i]$ and $x_i(R[\phi_i]) = R[\phi'_i]$, and consequently $[R[f'_i], R[\phi'_i]] = \Delta_{X'_i}$ for all i . \square

1.18. Regular pushouts in pointed Mal'cev categories with binary sums.

In a pointed category with binary sums and binary products, each pair of objects (X_1, X_2) defines a canonical comparison map $\theta_{X_1, X_2} : X_1 + X_2 \rightarrow X_1 \times X_2$, uniquely determined by the requirement that the composite morphism

$$X_i \twoheadrightarrow X_1 + X_2 \xrightarrow{\theta_{X_1, X_2}} X_1 \times X_2 \twoheadrightarrow X_j$$

is the identity (resp. the null morphism) if $i = j$ (resp. $i \neq j$), where $i, j \in \{1, 2\}$.

Recall that θ_{X_1, X_2} is a strong epimorphism for all objects X_1, X_2 precisely when the category is *unital* in the sense of the second author, and that every pointed Mal'cev category is unital, cf. [5, 7].

Note also that an exact Mal'cev category has coequalizers for reflexive pairs, so that an exact Mal'cev category with binary sums has all finite colimits. In order to shorten terminology, we call σ -pointed any pointed category with binary sums.

Later we shall need the following two examples of regular pushouts.

Proposition 1.19. *For any regular epimorphism $f : X \rightarrow Y$ and any object Z of a σ -pointed regular Mal'cev category, the following square*

$$\begin{array}{ccc} X + Z & \xrightarrow{\theta_{X, Z}} & X \times Z \\ f + Z \downarrow & & \downarrow f \times Z \\ Y + Z & \xrightarrow{\theta_{Y, Z}} & Y \times Z \end{array}$$

is a regular pushout.

Proof. The regular epimorphism $\theta_{R[f],Z} : R[f] + Z \twoheadrightarrow R[f] \times Z$ factors as below

$$\begin{array}{ccc} R[f] + Z & \xrightarrow{\theta_{R[f],Z}} & R[f] \times Z = R[f \times Z] \\ & \searrow & \nearrow \\ & R[f + Z] & \end{array}$$

inducing a regular epimorphism $R[f + Z] \rightarrow R[f \times Z]$ on the vertical kernel relations of the square above. Proposition 1.9 allows us to conclude. \square

Corollary 1.20. *For any objects X, Y, Z of a σ -pointed regular Mal'cev category, the following square*

$$\begin{array}{ccc} (X + Y) + Z & \xrightarrow{\theta_{X+Y,Z}} & (X + Y) \times Z \\ \theta_{X,Y} + Z \downarrow & & \downarrow \theta_{X,Y} \times Z \\ (X \times Y) + Z & \xrightarrow{\theta_{X \times Y,Z}} & (X \times Y) \times Z \end{array}$$

is a regular pushout.

1.21. Central subobjects, centres and centralizers. In a pointed Mal'cev category $(\mathbb{D}, \star_{\mathbb{D}})$, two morphisms with common codomain $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are said to *commute* [10, 43] if the square

$$\begin{array}{ccc} X \times Y & \xleftarrow{(\alpha_X, 1_Y)} & Y \\ (1_X, \alpha_Y) \uparrow & \searrow \phi_{f,g} & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

admits a (necessarily unique) filler $\phi_{f,g} : X \times Y \rightarrow Z$ making the whole diagram commute, where $\alpha_X : \star_{\mathbb{D}} \rightarrow X$ and $\alpha_Y : \star_{\mathbb{D}} \rightarrow Y$ denote the initial maps. A monomorphism $Z \rightarrowtail X$ which commutes with the identity $1_X : X \rightarrow X$ is called *central*, and the corresponding subobject is called a *central subobject* of X .

Every regular epimorphism $f : X \twoheadrightarrow Y$ with central kernel relation $R[f]$ has a central kernel $K[f]$. In pointed protomodular categories, the converse is true: the centrality of $K[f]$ implies the centrality of $R[f]$, so that central extensions are precisely the regular epimorphisms with central kernel, cf. [33, Proposition 2.2].

Recall [5, 6] that a pointed category is *protomodular* precisely when the category has pullbacks of split epimorphisms, and for each split epimorphism, section and kernel-inclusion form a strongly epimorphic cospan. Every finitely complete protomodular category is a Mal'cev category [5, Proposition 3.1.19]. The categories of groups and of Lie algebras are pointed protomodular. Moreover, in both categories, each object possesses a *centre*, i.e. a maximal central subobject. Central group (resp. Lie algebra) extensions are thus precisely regular epimorphisms $f : X \twoheadrightarrow Y$ with kernel $K[f]$ contained in the centre of X . This is of course the classical definition of a central extension in group (resp. Lie) theory.

In these categories, there exists more generally, for each subobject N of X , a so-called *centralizer*, i.e. a subobject $Z(N \rightarrowtail X)$ of X which is maximal among subobjects commuting with $N \rightarrowtail X$. The existence of centralizers has far-reaching categorical consequences, as shown by James Gray and the second author [14]. Since they are useful for our study of nilpotency, we discuss some of them here.

Following [7], we shall denote by $\text{Pt}_Z(\mathbb{D})$ the category of split epimorphisms in \mathbb{D} with fixed codomain Z . For each $f : Z \rightarrow Z'$ pulling back along f defines a functor $f^* : \text{Pt}_{Z'}(\mathbb{D}) \rightarrow \text{Pt}_Z(\mathbb{D})$ which we shall call *pointed base-change* along f , cf. Section 3.2. In particular, the terminal map $\omega_Z : Z \rightarrow 1_{\mathbb{D}}$ defines a functor $(\omega_Z)^* : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D})$. Since in a pointed regular Mal'cev category \mathbb{D} , morphisms commute if and only if their images commute, morphisms in $\text{Pt}_Z(\mathbb{D})$ of the form

$$\begin{array}{ccc} X \times Z & \xrightarrow{\phi_{f,f'}} & Y \\ p_Z \swarrow & & \searrow s \\ & Z & \end{array}$$

correspond bijectively to morphisms $f : X \rightarrow K[r]$ such that $X \xrightarrow{f} K[r] \rightarrow Y$ commutes with $s : Z \rightarrow Y$ in \mathbb{D} . Therefore, if split subobjects have centralizers in \mathbb{D} , then for each object Z , the functor $(\omega_Z)^* : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D}) : X \mapsto X \times Z$ admits a right adjoint $(\omega_Z)_* : \text{Pt}_Z(\mathbb{D}) \rightarrow \mathbb{D} : (r, s) \mapsto K[r] \cap Z(s)$.

A category with the property that for each object Z , the functor $(\omega_Z)^*$ has a right adjoint is called *algebraically cartesian closed* [14]. Algebraic cartesian closedness implies canonical isomorphisms $(X \times Z) +_Z (Y \times Z) \cong (X + Y) \times Z$ for all objects X, Y, Z , a property we shall call *algebraic distributivity*, cf. Section 5.8.

1.22. An algebraic Beck-Chevalley condition. –

The dual of an elementary topos is an exact Mal'cev category, cf. [18, Remark 5.8]. This suggests that certain diagram lemmas for elementary toposes admit a dual version in our algebraic setting. Supporting this analogy we establish here an “algebraic dual” of the well-known *Beck-Chevalley condition*. As a corollary we get a diagram lemma which will be used several times in Section 6.

Another instance of the same phenomenon is the *cogluing lemma* for regular epimorphisms in exact Mal'cev categories (cf. proof of Theorem 6.26a and Corollary 1.11) which is dual to a gluing lemma for monomorphisms in an elementary topos.

Lemma 1.23 (cf. Lemma 1.1 in [30]). *Consider a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{x} & X' & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{y} & Y' & \longrightarrow & Y'' \end{array}$$

in a regular category.

If the outer rectangle is a pullback and the left square is a regular pushout (1.8) then left and right squares are pullbacks.

Proof. The whole diagram contains three comparison maps: one for the outer rectangle, denoted $\phi : X \rightarrow Y \times_{Y''} X''$, one for the left and one for the right square, denoted respectively $\phi_l : X \rightarrow Y \times_{Y'} X'$ and $\phi_r : X' \rightarrow Y' \times_{Y''} X''$. We get the identity $\phi = y^*(\phi_r) \circ \phi_l$ where y^* denotes base-change along y . Since the outer rectangle is a pullback, ϕ is invertible so that ϕ_l is a section and $y^*(\phi_r)$ a retraction.

Since the left square is a regular pushout, the comparison map ϕ_l is a regular epimorphism and hence ϕ_l and $y^*(\phi_r)$ are both invertible. Since y is a regular epimorphism in a regular category, base-change y^* is conservative so that ϕ_r is invertible as well, i.e. both squares are pullbacks. \square

Proposition 1.24 (Algebraic Beck-Chevalley). *In an exact Mal'cev category with pushouts of split monomorphisms along regular epimorphisms, any pushout of regular epimorphisms*

$$\begin{array}{ccc} \bar{U} & \xrightarrow{\bar{g}} & \bar{V} \\ u \downarrow & & \downarrow v \\ U & \xrightarrow{g} & V \end{array}$$

yields a functor isomorphism $\bar{g}_! u^* \cong v^* g_!$ from the fibre over U to the fibre over \bar{V} .

Proof. We have to show that for any point (r, s) over U , the following diagram

$$\begin{array}{ccccc} \bar{U}' & \xrightarrow{\bar{g}'} & \bar{V}' & & \\ \bar{r} \nearrow & u' \searrow & \bar{r}' \nearrow & v' \searrow & \\ & U' & \xrightarrow{g'} & V' & \\ & \nearrow r & & \nearrow r' & \\ \bar{U} & \xrightarrow{\bar{g}} & \bar{V} & & \\ u \searrow & & v \searrow & & \\ & U & \xrightarrow{g} & V & \end{array}$$

in which $(\bar{r}, \bar{s}) = u^*(r, s)$ and $g_!(r, s) = (r', s')$ and $\bar{g}_!(\bar{r}, \bar{s}) = (\bar{r}', \bar{s}')$, has a right face which is a downward-oriented pullback; indeed, this amounts to the required identity $v^*(r', s') = (\bar{r}', \bar{s}')$.

Since bottom face and the upward-oriented front and back faces are pushouts, the top face is a pushout as well, which is regular by Corollary 1.11. Taking pullbacks in top and bottom faces induces a split epimorphism $U' \times_{V'} \bar{V}' \rightarrow U \times_V \bar{V}$ through which the left face of the cube factors as in the following commutative diagram

$$\begin{array}{ccccc} \bar{U}' & \xrightarrow{\quad} & U' \times_{V'} \bar{V}' & \xrightarrow{\quad} & U' \\ \bar{r} \downarrow & & \downarrow & & \downarrow r \\ \bar{U} & \xrightarrow{\quad} & U \times_V \bar{V} & \xrightarrow{\quad} & U \end{array}$$

in which the left square is a regular pushout by Corollary 1.10. Lemma 1.23 shows then that the right square is a pullback. Therefore, we get the following cube

$$\begin{array}{ccccc} U' \times_{V'} \bar{V}' & \xrightarrow{\quad} & \bar{V}' & & \\ \nearrow & & \bar{r}' \nearrow & v' \searrow & \\ & U' & \xrightarrow{g'} & V' & \\ & \nearrow r & & \nearrow r' & \\ U \times_V \bar{V} & \xrightarrow{\quad} & \bar{V} & & \\ \searrow & & v \searrow & & \\ & U & \xrightarrow{g} & V & \end{array}$$

in which the downward-oriented left face and the bottom face are pullbacks. Therefore, the composite of the top face followed by the downward-oriented right face is a pullback. Moreover, as above, the top face is a regular pushout. It follows then from Lemma 1.23 that the downward-oriented right face is a pullback as required. \square

Corollary 1.25. *In an exact Mal'cev category with pushouts of split monomorphisms along regular epimorphisms, each commuting square of natural transformations of split epimorphisms*

$$\begin{array}{ccccc}
 \bar{U}' & \xrightarrow{\bar{g}'} & \bar{V}' & & \\
 \bar{f} \swarrow & u' \searrow & \bar{f}' \swarrow & v' \searrow & \\
 & U' & \xrightarrow{g'} & V' & \\
 \bar{U} \xrightarrow{\quad} & \downarrow f & \bar{V} & \downarrow f' & \\
 u \searrow & U & \xrightarrow{g} & V & \\
 & \downarrow \bar{g} & & \downarrow v &
 \end{array}$$

such that all horizontal arrows are regular epimorphisms, and front and back faces are upward-oriented pushouts, induces the following upper pushout square

$$\begin{array}{ccc}
 \bar{U}' & \xrightarrow{\bar{g}'} & \bar{V}' \\
 \bar{f} \swarrow & (\bar{f}, u') \searrow & \downarrow \bar{f}' \\
 & \bar{U} \times_U U' & \xrightarrow{\bar{g} \times g'} \bar{V} \times_V V' \\
 & \downarrow \bar{g} & \downarrow v' \\
 \bar{U} & \xrightarrow{\bar{g}} & \bar{V}
 \end{array}$$

in which the kernel relation of the regular epimorphism $(\bar{f}', v') : \bar{V}' \rightarrow \bar{V} \times_V V'$ may be identified with the intersection $R[\bar{f}'] \cap R[v']$.

Proof. Taking downward-oriented pullbacks in left and right face of the first diagram yields precisely a diagram as studied in the proof of Proposition 1.24. This implies that the front face of the second diagram is an upward-oriented pushout. Since the back face of the second diagram is also an upward-oriented pushout, the upper square is a pushout as well, as asserted. Since $(\bar{f}', v') : \bar{V}' \rightarrow \bar{V} \times_V V'$ is the comparison map of a regular pushout by Corollary 1.10, its kernel relation is the intersection of the kernel relations of \bar{f}' and of v' . \square

2. AFFINE OBJECTS AND NILPOTENCY

Definition 2.1. *Let \mathbb{C} be a full subcategory of a Mal'cev category \mathbb{D} .*

An object X of \mathbb{D} is said to be \mathbb{C} -affine if there exists a morphism $f : X \rightarrow Y$ in \mathbb{D} with central kernel relation $R[f]$ and with codomain Y in \mathbb{C} .

The morphism f is called a \mathbb{C} -nilindex for X .

We shall write $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ for the full replete subcategory of \mathbb{D} spanned by the \mathbb{C} -affine objects of \mathbb{D} . Clearly $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ contains \mathbb{C} and is a Mal'cev category.

When \mathbb{C} consists only of a terminal object $1_{\mathbb{D}}$ of \mathbb{D} , we call the \mathbb{C} -affine objects simply the *affine objects* of \mathbb{D} and write $\text{Aff}_{\mathbb{C}}(\mathbb{D}) = \text{Aff}(\mathbb{D})$. Recall that the unique morphism $X \rightarrow 1_{\mathbb{D}}$ has a central kernel relation precisely when the indiscrete equivalence relation on X centralizes itself, which amounts to the existence of a (necessarily unique associative and commutative) *Mal'cev operation* on X .

When \mathbb{D} is *pointed*, such a Mal'cev operation on X induces (and is induced by) an *abelian group* structure on X . For a pointed Mal'cev category \mathbb{D} , the category $\text{Aff}(\mathbb{D})$ of affine objects is thus the category $\text{Ab}(\mathbb{D})$ of abelian group objects of \mathbb{D} .

Remark 2.2. When \mathbb{D} is a *regular* Mal'cev category, any nilindex $f : X \rightarrow Y$ factors as a regular epimorphism $\tilde{f} : X \twoheadrightarrow f(X)$ followed by a monomorphism $f(X) \hookrightarrow Y$ with codomain in \mathbb{C} ; therefore, if \mathbb{C} is *closed under taking subobjects* in \mathbb{D} , this defines a strongly epimorphic nilindex \tilde{f} for X with same central kernel relation as f . In other words, for regular Mal'cev categories \mathbb{D} and subcategories \mathbb{C} which are closed under taking subobjects in \mathbb{D} , the \mathbb{C} -affine objects of \mathbb{D} are precisely the objects which are obtained as central extensions of objects of \mathbb{C} .

Proposition 2.3. *For any full subcategory \mathbb{C} of a Mal'cev category \mathbb{D} , the subcategory $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ is closed under taking subobjects in \mathbb{D} . If \mathbb{C} is closed under taking binary products in \mathbb{D} then $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ as well, so that $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ is finitely complete.*

Proof. Let $m : X \hookrightarrow X'$ be a monomorphism with \mathbb{C} -affine codomain X' . If $f' : X' \rightarrow Y'$ is a nilindex for X' , then $f m : X \rightarrow Y'$ is a nilindex for X , since central equivalence relations are stable under pointed base-change along monomorphisms and we have $R[f m] = m^{-1}(R[f'])$.

If X and Y are \mathbb{C} -affine with nilindices f and g then $f \times g$ is a nilindex for $X \times Y$ since maps with central kernel relations are stable under products. \square

A *Birkhoff subcategory* [44] of a regular category \mathbb{D} is a subcategory \mathbb{C} which is closed under taking subobjects, products and quotients in \mathbb{D} . A Birkhoff subcategory of an exact (resp. Mal'cev) category is exact (resp. Mal'cev), and regular epimorphisms in \mathbb{C} are those morphisms in \mathbb{C} which are regular epimorphisms in \mathbb{D} .

Proposition 2.4. *Let \mathbb{C} be a full subcategory of an exact Mal'cev category \mathbb{D} .*

If \mathbb{C} is closed under taking subobjects and quotients in \mathbb{D} then $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ as well. In particular, if \mathbb{C} is a Birkhoff subcategory of \mathbb{D} , then $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ as well.

Proof. Let X be a \mathbb{C} -affine object of \mathbb{D} with nilindex $f : X \rightarrow Y$. We can suppose f is a regular epimorphism, cf. Remark 2.2. Thanks to Proposition 2.4 it remains to establish closure under quotient. Let $g : X \twoheadrightarrow X'$ be a regular epimorphism in \mathbb{D} . Since \mathbb{D} is exact, the pushout of f along g exists in \mathbb{D}

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

and f' is a central extension since f is, cf. Corollary 1.11. By hypothesis \mathbb{C} is stable under quotients. Therefore the quotient Y' belongs to \mathbb{C} , and f' is a nilindex for X' so that X' is \mathbb{C} -affine as required. \square

2.5. The \mathbb{C} -lower central sequence. The previous definition is clearly the beginning of an iterative process. We write $\mathbb{C} = \text{Nil}_{\mathbb{C}}^0(\mathbb{D})$ and define inductively $\text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ to be the category $\text{Aff}_{\text{Nil}_{\mathbb{C}}^{n-1}(\mathbb{D})}(\mathbb{D})$. The objects of this category $\text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ are called

the \mathbb{C} -nilpotent objects of order n of \mathbb{D} , and we get the following diagram

$$\begin{array}{c} \mathbb{D} \\ \swarrow \quad \uparrow \quad \searrow \\ \mathbb{C} \longrightarrow \text{Nil}_{\mathbb{C}}^1(\mathbb{D}) \longrightarrow \text{Nil}_{\mathbb{C}}^2(\mathbb{D}) \cdots \cdots \text{Nil}_{\mathbb{C}}^n(\mathbb{D}) \longrightarrow \text{Nil}_{\mathbb{C}}^{n+1}(\mathbb{D}) \cdots \cdots \end{array}$$

which we call the \mathbb{C} -lower central sequence of \mathbb{D} .

If $\mathbb{C} = \{1_{\mathbb{D}}\}$, we obtain the (absolute) lower central sequence of \mathbb{D} :

$$\begin{array}{c} \mathbb{D} \\ \swarrow \quad \uparrow \quad \searrow \\ \{1_{\mathbb{D}}\} \longrightarrow \text{Nil}^1(\mathbb{D}) \longrightarrow \text{Nil}^2(\mathbb{D}) \cdots \cdots \text{Nil}^n(\mathbb{D}) \longrightarrow \text{Nil}^{n+1}(\mathbb{D}) \cdots \cdots \end{array}$$

Remark 2.6. It follows from Remark 2.2 and an iterative application of Proposition 2.4 that for an exact Mal'cev category \mathbb{D} , the nilpotent objects of order n are precisely those which can be obtained as an n -fold central extension of the terminal object $1_{\mathbb{D}}$ and that moreover $\text{Nil}^n(\mathbb{D})$ is a Birkhoff subcategory of \mathbb{D} .

If \mathbb{D} is the category of groups (resp. Lie algebras) then $\text{Nil}^n(\mathbb{D})$ is precisely the full subcategory spanned by nilpotent groups (resp. Lie algebras) of class $\leq n$. Indeed, it is well-known that a group (resp. Lie algebra) is nilpotent of class $\leq n$ precisely when it can be obtained as an n -fold “central extension” of the trivial group (resp. Lie algebra), and we have seen in Section 1.21 that the group (resp. Lie) theorist’s definition of central extension coincides with ours’.

Definition 2.7. A Mal'cev category \mathbb{D} with full subcategory \mathbb{C} is called \mathbb{C} -nilpotent of order n (resp. of class n) if $\mathbb{D} = \text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ (resp. if n is the least such integer).

When $\mathbb{C} = \{1_{\mathbb{D}}\}$ the prefix \mathbb{C} will be dropped, and instead of “nilpotent of order n ” we also just say “ n -nilpotent”.

Proposition 2.8. A Mal'cev category is n -nilpotent if and only if each morphism is n -fold centrally decomposable.

A regular Mal'cev category is n -nilpotent if and only if each morphism factors as an n -fold central extension followed by a monomorphism.

Proof. The second statement follows from the first by Lemma 1.3. If each morphism is n -fold centrally decomposable, then this holds for terminal maps $\omega_X : X \rightarrow 1_{\mathbb{D}}$, so that all objects are n -nilpotent. Conversely, assume that all objects are n -nilpotent, i.e. that for all objects X , the terminal map ω_X is n -fold centrally decomposable. Then, for each morphism $f : X \rightarrow Y$, the identity $\omega_X = \omega_Y f$ together with Lemma 1.5 imply that f is n -fold centrally decomposable as well. \square

2.9. Epireflections, Birkhoff reflections and central reflections. –

We shall see that if \mathbb{C} is a reflective subcategory of \mathbb{D} , then the categories $\text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ are again reflective subcategories of \mathbb{D} , provided \mathbb{D} and the reflection fulfill suitable conditions. In order to give precise statements we need to fix some terminology.

A full replete subcategory \mathbb{C} of \mathbb{D} is called *reflective* if the inclusion $\mathbb{C} \hookrightarrow \mathbb{D}$ admits a left adjoint functor $I : \mathbb{D} \rightarrow \mathbb{C}$, called *reflection*. The unit of the adjunction at

an object X of \mathbb{D} will be denoted by $\eta_X : X \rightarrow I(X)$. Reflective subcategories \mathbb{C} are stable under formation of limits in \mathbb{D} . In particular, reflective subcategories of Mal'cev categories are Mal'cev categories.

A reflective subcategory \mathbb{C} of \mathbb{D} is called *epireflective* if \mathbb{C} is closed under taking subobjects in \mathbb{D} . Epireflective subcategories are characterized by the property that the unit $\eta_X : X \rightarrow I(X)$ is pointwise a *strong* epimorphism. Epireflective subcategories of regular categories are regular categories.

A *Birkhoff reflection* (cf. [15]) is an epireflection $I : \mathbb{D} \rightarrow \mathbb{C}$ such that for each regular epimorphism $f : X \rightarrow Y$ in \mathbb{D} , the following naturality square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & I(X) \\ f \downarrow & & \downarrow I(f) \\ Y & \xrightarrow{\eta_Y} & I(Y) \end{array}$$

is a *regular pushout* (see Section 1.8 and Proposition 1.9).

A subcategory of \mathbb{D} defined by a Birkhoff reflection is a Birkhoff subcategory of \mathbb{D} , and is thus exact whenever \mathbb{D} is. It follows from Corollary 1.11 that a reflective subcategory of an exact Mal'cev category is a Birkhoff subcategory *if and only if* the reflection is a Birkhoff reflection.

A *central reflection* is an epireflection $I : \mathbb{D} \rightarrow \mathbb{C}$ with the property that the unit $\eta_X : X \rightarrow I(X)$ is pointwise a central extension.

The following exactness result, due to Diana Rodelo and the second author [15], will be used at several places later on.

Proposition 2.10. *In a regular Mal'cev category, epireflections preserve pullback squares of split epimorphisms, and Birkhoff reflections preserve pullbacks of split epimorphisms along regular epimorphisms.*

Proof. See Proposition 3.4 and Theorem 3.16 in [15]. \square

Lemma 2.11. *Let \mathbb{C} be a reflective subcategory of \mathbb{D} with reflection I , and assume that $\eta_X : X \rightarrow I(X)$ factors through an epimorphism $f : X \twoheadrightarrow Y$ as in:*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & I(X) \\ f \downarrow & \nearrow \eta & \\ Y & & \end{array}$$

Then $I(f)$ is an isomorphism and we have $\eta = I(f)^{-1}\eta_Y$.

Proof. Consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & I(X) \\ f \downarrow & \nearrow \eta & \downarrow I(f) \\ Y & \xrightarrow{\eta_Y} & I(Y) \end{array}$$

where the lower triangle commutes because f is an epimorphism. If we apply the reflection I to the whole diagram we get two horizontal isomorphisms $I(\eta_X)$ and $I(\eta_Y)$. It follows that $I(\eta)$ is an isomorphism as well, hence so is $I(f)$, and $\eta = I(f)^{-1}\eta_Y$. \square

Lemma 2.12. *For any reflective subcategory \mathbb{C} of a Mal'cev category \mathbb{D} the \mathbb{C} -affine objects of \mathbb{D} are those X for which the unit η_X has a central kernel relation.*

Proof. If $\eta_X : X \rightarrow I(X)$ has a central kernel relation then X is \mathbb{C} -affine. Conversely, let X be \mathbb{C} -affine with nilindex $f : X \rightarrow Y$. Then Y is an object of the reflective subcategory \mathbb{C} so that f factors through $\eta_X : X \rightarrow I(X)$. Accordingly, we get $R[\eta_X] \subset R[f]$, and hence $R[\eta_X]$ is central because $R[f]$ is. \square

Corollary 2.13. *An epireflection $I : \mathbb{D} \rightarrow \mathbb{C}$ of a regular Mal'cev category \mathbb{D} is central if and only if \mathbb{D} is \mathbb{C} -nilpotent of order 1 (i.e. all objects of \mathbb{D} are \mathbb{C} -affine).*

Theorem 2.14 (cf. [43], Section 4.3, and [26], Proposition 7.8). –

For a reflective subcategory \mathbb{C} of a finitely cocomplete regular Mal'cev category \mathbb{D} , the category $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ is an epireflective subcategory of \mathbb{D} .

The associated epireflection $I_{\mathbb{C}}^1 : \mathbb{D} \rightarrow \text{Aff}_{\mathbb{C}}(\mathbb{D})$ is obtained by factoring the unit $\eta_X : X \rightarrow I(X)$ universally through a map with central kernel relation.

If \mathbb{C} is a reflective Birkhoff subcategory of a finitely cocomplete exact Mal'cev category \mathbb{D} , then the epireflection $I_{\mathbb{C}}^1 : \mathbb{D} \rightarrow \text{Aff}_{\mathbb{C}}(\mathbb{D})$ is a Birkhoff reflection.

Proof. Proposition 1.6 yields the following factorization of the unit:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & I(X) \\ \eta_X^1 \downarrow & \nearrow \bar{\eta}_X & \\ I_{\mathbb{C}}^1(X) & & \end{array}$$

Since $\bar{\eta}_X$ has a central kernel relation and $I(X)$ is an object of \mathbb{C} , the object $I_{\mathbb{C}}^1(X)$ belongs to $\text{Aff}_{\mathbb{C}}(\mathbb{D})$. We claim that the maps $\eta_X^1 : X \rightarrow I_{\mathbb{C}}^1(X)$ have the universal property of the unit of an epireflection $I_{\mathbb{C}}^1 : \mathbb{D} \rightarrow \text{Aff}_{\mathbb{C}}(\mathbb{D})$.

Let $f : X \rightarrow T$ be a map with \mathbb{C} -affine codomain T which means that η_T has a central kernel relation. Then consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & T & & \\ \eta_X^1 \searrow & & \downarrow & \text{=} & \\ & I_{\mathbb{C}}^1(X) & \xrightarrow{\bar{f}} & T & \\ \eta_X \downarrow & \nearrow \bar{\eta}_X & & \downarrow \eta_T & \\ I(X) & \xrightarrow{I(f)} & I(T) & & \end{array}$$

According to Proposition 1.6, there is a unique factorization \bar{f} making the diagram commute. If \mathbb{D} is exact and \mathbb{C} a Birkhoff subcategory, the subcategory $\text{Aff}_{\mathbb{C}}(\mathbb{D})$ is closed under taking subobjects and quotients by Proposition 2.4. The reflection $I_{\mathbb{C}}^1$ is thus a Birkhoff reflection in this case. \square

Remark 2.15. A reflective Birkhoff subcategory \mathbb{C} of a semi-abelian category \mathbb{D} satisfies all hypotheses of the preceding theorem. In this special case, the Birkhoff reflection $I_{\mathbb{C}}^1 : \mathbb{D} \rightarrow \text{Aff}_{\mathbb{C}}(\mathbb{D})$ is given by the formula

$$I_{\mathbb{C}}^1(X) = X/[X, K[\eta_X]]$$

where $[X, K[\eta_X]]$ is the Huq commutator of X and $K[\eta_X]$, cf. Section 1.21.

Indeed, the pointed protomodularity of \mathbb{D} implies that the kernel of the quotient map $X \rightarrow X/[\nabla_X, R[\eta_X]]$ is canonically isomorphic to the Huq commutator $[X, K[\eta_X]]$ so that the formula above follows from Proposition 1.6.

2.16. The Birkhoff nilpotency tower. According to Theorem 2.14, any reflective Birkhoff subcategory \mathbb{C} of a finitely cocomplete exact Mal'cev category \mathbb{D} produces iteratively the following commutative diagram of Birkhoff reflections:

$$\begin{array}{c} \mathbb{D} \\ \swarrow I \quad \searrow I_{\mathbb{C}}^1 \quad \downarrow I_{\mathbb{C}}^2 \quad \searrow I_{\mathbb{C}}^n \quad \searrow I_{\mathbb{C}}^{n+1} \\ \mathbb{C} \leftarrow \text{Nil}_{\mathbb{C}}^1(\mathbb{D}) \leftarrow \text{Nil}_{\mathbb{C}}^2(\mathbb{D}) \cdots \cdots \text{Nil}_{\mathbb{C}}^n(\mathbb{D}) \leftarrow \text{Nil}_{\mathbb{C}}^{n+1}(\mathbb{D}) \cdots \cdots \end{array}$$

A Birkhoff subcategory of an exact Mal'cev category is an exact Mal'cev category so that the subcategories $\text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ are all exact Mal'cev categories, and the horizontal reflections $\text{Nil}_{\mathbb{C}}^{n+1}(\mathbb{D}) \rightarrow \text{Nil}_{\mathbb{C}}^n(\mathbb{D})$ are *central* reflections by Corollary 2.13.

In the special case $\mathbb{C} = \{1_{\mathbb{D}}\}$ we get the following commutative diagram of Birkhoff subcategories and Birkhoff reflections:

$$\begin{array}{c} \mathbb{D} \\ \swarrow I \quad \searrow I^1 \quad \downarrow I^2 \quad \searrow I^n \quad \searrow I^{n+1} \\ \{1_{\mathbb{D}}\} \leftarrow \text{Nil}^1(\mathbb{D}) \leftarrow \text{Nil}^2(\mathbb{D}) \cdots \cdots \text{Nil}^n(\mathbb{D}) \leftarrow \text{Nil}^{n+1}(\mathbb{D}) \cdots \cdots \end{array}$$

If \mathbb{D} is pointed, then the first Birkhoff reflection $I^1 = I_{\{1_{\mathbb{D}}\}}^1 : \mathbb{D} \rightarrow \text{Nil}^1(\mathbb{D})$ can be identified with the classical abelianization functor $\mathbb{D} \rightarrow \text{Ab}(\mathbb{D})$. In particular, the abelian group objects of \mathbb{D} are precisely the nilpotent objects of order 1.

When \mathbb{C} is a reflective Birkhoff subcategory of a finitely cocomplete exact Mal'cev category \mathbb{D} , then \mathbb{D} is \mathbb{C} -nilpotent of class n if and only if n is the least integer such that either the unit of the n -th Birkhoff reflection $I_{\mathbb{C}}^n$ is invertible, or equivalently, the $(n-1)$ st Birkhoff reflection $I_{\mathbb{C}}^{n-1}$ is a *central* reflection, see Corollary 2.13.

A finite limit and finite colimit preserving functor is called *exact*. A functor between exact Mal'cev categories with binary sums is exact if and only if it preserves finite limits, regular epimorphisms and binary sums, cf. Section 1.18.

Lemma 2.17. *Any exact functor $F : \mathbb{D} \rightarrow \mathbb{E}$ between exact Mal'cev categories with binary sums commutes with the n -th Birkhoff reflections, i.e. $I_{\mathbb{E}}^n \circ F \cong F|_{\text{Nil}^n(\mathbb{D})} \circ I_{\mathbb{D}}^n$.*

Proof. According to Theorem 2.14 and Proposition 1.6 it suffices to show that F takes the canonical factorization of $f : X \rightarrow Y$ through the central extension $\zeta_f : X/[\nabla_X, R[f]] \rightarrow Y$ to the factorization of $F(f) : F(X) \rightarrow F(Y)$ through the central extension $\zeta_{F(f)} : F(X)/[\nabla_{F(X)}, R[F(f)]] \rightarrow F(Y)$. Since F is left exact, we have $F(\nabla_X) = \nabla_{F(X)}$ and $F(R[f]) = R[F(f)]$, and since F preserves regular epimorphisms, we have $F(X/[\nabla_X, R[f]]) = F(X)/F([\nabla_X, R[f]])$. It remains to be shown that F preserves Smith commutators. This follows from exactness of F and the fact that in a finitely cocomplete exact Mal'cev category the Smith commutator is given by an explicit formula involving only finite limits and finite colimits. \square

3. AFFINE MORPHISMS AND CENTRAL REFLECTIONS

In this section we establish a useful property of central reflections in exact Mal'cev categories, namely that the unit of a central reflection is actually pointwise an affine extension. Since this property might be useful in other contexts as well, we first discuss possible weakenings of the notion of exactness.

3.1. Quasi-exact and efficiently regular Mal'cev categories. –

An exact category is a regular category in which every equivalence relation is *effective*, i.e. the kernel relation of its quotient map. In general, effective equivalence relations R on X have the property that the inclusion $R \rightarrow X \times X$ is a strong monomorphism. Equivalence relations (EQR) with this property are called *strong*. A regular category with effective strong equivalence relations is called *quasi-exact*.

Any quasi-topos (cf. Penon [57]) is quasi-exact so that there are plenty examples of quasi-exact categories which are not exact. There are also quasi-exact Mal'cev categories which are not exact, as for instance the category of topological groups and continuous group homomorphisms. Further weakenings of exactness occur quite naturally as shown in the following chain of implications:

$$\text{exact} \implies \text{quasi-exact} \implies \text{efficiently regular} \implies \text{fibrational effective EQR}$$

A category is called *efficiently regular* [15] if every equivalence relation (X, S) , which is a *regular refinement* of an effective equivalence relation (X, R) , is itself effective. By regular refinement we mean any map of equivalence relations $(X, S) \rightarrow (X, R)$ inducing the identity on X and a regular monomorphism $S \rightarrow R$. An effective equivalence relation (X, R) is called *fibrational* if each cartesian map of equivalence relations $(Y, S) \rightarrow (X, R)$ has an effective equivalence relation as domain. According to Janelidze-Sobral-Tholen [48] an effective equivalence relation (X, R) is fibrational if and only if its quotient map $X \rightarrow X/R$ is an *effective descent morphism*, i.e. base-change along $X \rightarrow X/R$ is a monadic functor.

The second implication above follows from the fact that any regular monomorphism is a strong monomorphism, while the third implication follows from the facts that for any cartesian map of equivalence relations $f : (Y, S) \rightarrow (X, R)$ the induced map on relations $S \rightarrow f^*(R)$ is a regular (even split) monomorphism, and that in any regular category, effective equivalence relations are closed under inverse image.

3.2. Fibration of points and essentially affine categories. –

Recall [7] that for any category \mathbb{D} , we denote by $\text{Pt}(\mathbb{D})$ the category whose objects are the split epimorphisms (“generalized points”) of \mathbb{D} and whose morphisms are the commuting squares between such split epimorphisms, and that $\P_{\mathbb{D}} : \text{Pt}(\mathbb{D}) \rightarrow \mathbb{D}$ denotes the functor associating to each split epimorphism its codomain.

The functor $\P_{\mathbb{D}} : \text{Pt}(\mathbb{D}) \rightarrow \mathbb{D}$ is a fibration (the so-called *fibration of points*) whenever \mathbb{D} has pullbacks of split epimorphisms. The $\P_{\mathbb{D}}$ -cartesian maps are precisely pullbacks of split epimorphisms. Given any morphism $f : X \rightarrow Y$ in \mathbb{D} , base-change along f with respect to the fibration $\P_{\mathbb{D}}$ is denoted by $f^* : \text{Pt}_Y(\mathbb{D}) \rightarrow \text{Pt}_X(\mathbb{D})$, and will be called *pointed base-change* in order to distinguish it from the classical base-change $\mathbb{D}/Y \rightarrow \mathbb{D}/X$ on slices.

Pointed base-change f^* has a *left adjoint* pointed cobase-change $f_!$ if and only if pushouts along f of split monomorphisms with domain X exist in \mathbb{D} . In this case pointed cobase-change along f is given by precisely this pushout, cf. [6].

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \uparrow s & & \uparrow s' \\ X & \xrightarrow{f} & Y \end{array}$$

be the indiscrete equivalence relation on X with section $s_0^X: X \rightarrow X \times X$. Since pointed base-change f^* is an equivalence of categories, there is a split epimorphism

$(r, s) : Y' \rightrightarrows Y$ such that $f^*(r, s) = (p_0^X, s_0^X)$ and we get the right hand pullback of diagram

$$\begin{array}{ccccc}
 R[\check{f}] & \xrightleftharpoons[p_0]{\check{p}_1} & X \times X & \xrightarrow{\check{f}} & Y' \\
 \downarrow R(p_0^X, r) & & \downarrow p_0^X & & \downarrow r \\
 R[s_0^X, s] & \xrightleftharpoons[p_1]{\check{p}_0} & X & \xrightarrow{f} & Y \\
 \downarrow R(p_0^X, r) & & \downarrow p_0^X & & \downarrow r \\
 R[f] & \xrightleftharpoons[p_0]{p_1} & X & \xrightarrow{f} & Y
 \end{array}$$

in which the left hand side are the respective kernel relations. Therefore the left hand side consists of two pullbacks, and the map $p_1^X \check{p}_0$ produces the required connector between $R[f]$ and the indiscrete equivalence relation ∇_X on X . \square

In particular, if an epireflection $I : \mathbb{D} \rightarrow \mathbb{C}$ of a Mal'cev category \mathbb{D} has a pointwise $\P_{\mathbb{D}}$ -affine unit $\eta_X : X \rightarrow I(X)$, then it is a central reflection. The following converse will be essential in understanding nilpotency.

Theorem 3.5. *A central reflection I of an efficiently regular Mal'cev category \mathbb{D} has a unit which is pointwise an affine extension. In particular, morphisms $f : X \rightarrow Y$ with invertible image $I(f) : I(X) \rightarrow I(Y)$ are necessarily $\P_{\mathbb{D}}$ -affine.*

Proof. The second assertion follows from the first since $\P_{\mathbb{D}}$ -affine morphisms fulfill the two-out-of-three property and isomorphisms are $\P_{\mathbb{D}}$ -affine.

For the first assertion note that in a regular Mal'cev category pointed base-change along a regular epimorphism is fully faithful, and hence $\eta_Y^* : \text{Pt}_{I(Y)}(\mathbb{D}) \rightarrow \text{Pt}_Y(\mathbb{D})$ is a fully faithful functor. Corollary 3.9 below shows that η_Y^* is essentially surjective, hence η_Y^* is an equivalence of categories for all objects Y in \mathbb{D} . \square

3.6. Centralizing double relations. Given a pair (R, S) of equivalence relations on Y , we denote by $R \square S$ the inverse image of the reflexive relation $S \times S$ under $(p_0^R, p_1^R) : R \rightarrow Y \times Y$. This defines a double relation

$$\begin{array}{ccc}
 R \square S & \xrightleftharpoons[p_0^S]{q_1^S} & S \\
 \downarrow q_0^R & & \downarrow p_0^S \\
 R & \xrightleftharpoons[p_1^R]{q_1^R} & Y \\
 \downarrow q_0^R & & \downarrow p_0^S \\
 R & \xrightleftharpoons[p_0^R]{p_1^R} & Y
 \end{array}$$

actually the largest double relation relating R and S . In set-theoretical terms, this double relation $R \square S$ corresponds to the subset of elements (u, v, u', v') of Y^4 such that the quadratic set of relations $uRu', vRv', uSv, u'Sv'$ holds.

Lemma 3.7. *Any split epimorphism $(r, s) : X \rightrightarrows Y$ of a regular Mal'cev category with epireflection I and unit η induces the following diagram*

$$\begin{array}{ccccc}
 R[\eta_{R[r]}] & \rightleftarrows & R[r] & \xrightarrow{\eta_{R[r]}} & I(R[r]) \\
 \downarrow R(p_0^r, Ip_0^r) & & \downarrow p_0^r & & \downarrow Ip_0^r \\
 R[\eta_X] & \rightleftarrows & X & \xrightarrow{\eta_X} & IX \\
 \downarrow R(r, Ir) & & \downarrow r & & \downarrow Ir \\
 R[\eta_Y] & \rightleftarrows & Y & \xrightarrow{\eta_Y} & IY
 \end{array}$$

$\begin{array}{ccc} R(p_1^r, Ip_1^r) & p_1^r & Ip_1^r \\ R(s, Is) & s & Is \end{array}$

in which the rows and the two right vertical columns represent kernel relations. The left most column represents then the kernel relation of the induced map $R(r, Ir) : R[\eta_X] \rightarrow R[\eta_Y]$, and we have $R[\eta_{R[r]}] = R[\eta_X] \square R[r] = R[R(r, Ir)]$.

Proof. This follows from Proposition 2.10 which shows that $I(R[r])$ may be identified with the kernel relation $R[I(r)]$ of the split epimorphism $Ir : IX \rightarrow IY$. \square

For sake of simplicity a split epimorphism $(r, s) : X \rightrightarrows Y$ is called a \mathbb{C} -affine point over Y whenever its domain X is \mathbb{C} -affine.

Proposition 3.8. *Let \mathbb{C} be a reflective subcategory of an efficiently regular Mal'cev category \mathbb{D} with reflection I and unit η .*

Any \mathbb{C} -affine point (r, s) over Y is the image under η_Y^ of a \mathbb{C} -affine point (\bar{r}, \bar{s}) over IY such that both points have isomorphic reflections in \mathbb{C} .*

Proof. Since by Lemma 2.12 η_X has a central kernel relation, the kernel relations $R[\eta_X]$ and $R[r]$ centralize each other. This induces a centralizing double relation

$$\begin{array}{ccc}
 R[\eta_X] \times_X R[r] & \rightleftarrows & R[r] \\
 \downarrow \vdots & & \downarrow p_0^r \\
 R[\eta_X] & \rightleftarrows & X
 \end{array}$$

$\downarrow p_1^r$

which we consider as a cartesian split epimorphism of equivalence relations (disregarding the dotted arrows). Pulling back along the monomorphism

$$\begin{array}{ccc}
 R[\eta_X] & \rightleftarrows & X \\
 \uparrow R(s, Is) & & \uparrow s \\
 R[\eta_Y] & \rightleftarrows & Y
 \end{array}$$

yields on the left hand side of the following diagram

$$\begin{array}{ccccc}
 R_I[(r, s)] & \rightleftarrows & X & \xrightarrow{\quad q \quad} & \bar{X} \\
 \updownarrow & & \updownarrow \scriptstyle r, s & & \updownarrow \scriptstyle \bar{r}, \bar{s} \\
 R[\eta_Y] & \rightleftarrows & Y & \xrightarrow{\quad \eta_Y \quad} & IY
 \end{array}$$

another cartesian split epimorphism of equivalence relations. Since \mathbb{D} is efficiently regular, the upper equivalence relation is effective with quotient $q : X \twoheadrightarrow \bar{X}$. We claim that the induced point $(\bar{r}, \bar{s}) : \bar{X} \rightrightarrows IY$ has the required properties.

Indeed, the right square is a pullback by a well-known result of Barr and Kock, cf. [5, Lemma A.5.8], so that $\eta_Y^*(\bar{r}, \bar{s}) = (r, s)$. The centralizing double relation $R[\eta_X] \times_X R[r]$ is coherently embedded in the double relation $R[\eta_X] \square R[r]$ of Lemma 3.7, cf. [5, Proposition 2.6.13]. This induces an inclusion $R_I[(r, s)] \hookrightarrow R[\eta_X]$ and hence a morphism $\phi : \bar{X} \twoheadrightarrow IX$ such that $\phi q = \eta_X$.

According to Lemma 2.11, we get an isomorphism $Iq : IX \cong I\bar{X}$ compatible with the units. The kernel relation $R[\eta_{\bar{X}}]$ is thus the direct image of the central kernel relation $R[\eta_X]$ under the regular epimorphism $q : X \rightarrow \bar{X}$ and as such central as well. In particular, (\bar{r}, \bar{s}) is a \mathbb{C} -affine point with same reflection in \mathbb{C} as (r, s) . \square

Corollary 3.9. *For an efficiently regular Mal'cev category \mathbb{D} with central reflection I and unit η , pointed base-change $\eta_Y^* : \text{Pt}_{I(Y)}(\mathbb{D}) \rightarrow \text{Pt}_Y(\mathbb{D})$ is essentially surjective.*

Proof. Since the reflection is central, Corollary 2.13 shows that Proposition 3.8 applies to the whole fibre $\text{Pt}_Y(\mathbb{D})$ whence essential surjectivity of η_Y^* . \square

3.10. Affine extensions in efficiently regular Mal'cev categories. –

A functor $G : \mathbb{E} \rightarrow \mathbb{E}'$ is called *saturated on quotients* if for each object A in \mathbb{E} and each strong epimorphism $g' : G(A) \rightarrow B'$ in \mathbb{E}' , there exists a strong epimorphism $g : A \rightarrow B$ in \mathbb{E} such that $G(g)$ and g' are isomorphic under $G(A)$. Note that a right adjoint functor $G : \mathbb{E} \rightarrow \mathbb{E}'$ is essentially surjective whenever it is saturated on quotients and each object B' of \mathbb{E}' is the quotient of an object B'' for which the unit $\eta_{B''} : B'' \rightarrow GF(B'')$ is invertible.

Lemma 3.11. *In an efficiently regular Mal'cev category, pointed base-change along a regular epimorphism is saturated on quotients.*

Proof. Let $f : X \twoheadrightarrow Y$ be a regular epimorphism, let (r, s) be a point over Y , and $l : f^*((r, s)) \twoheadrightarrow (r', s')$ be a quotient map over X . Consider the following diagram

$$\begin{array}{ccccc}
 R[f'] & \rightleftarrows & X' & \xrightarrow{f'} & Y' \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 & & S & \rightleftarrows & X'' \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 R[f] & \rightleftarrows & X & \xrightarrow{f} & Y
 \end{array}$$

with additional arrows: $p_1^{f'} : R[f'] \rightarrow X'$, $p_0^{f'} : R[f'] \rightarrow S$, $p_1^f : R[f] \rightarrow X$, $p_0^f : R[f] \rightarrow S$, $l : X' \rightarrow S$, $f^*(r) : S \rightarrow X$, $f^*(s) : S \rightarrow X$, $r' : X' \rightarrow X$, $s' : X' \rightarrow X$, $f'' : X'' \rightarrow Y''$, $\bar{l} : Y' \rightarrow Y''$, $\rho : Y \rightarrow Y''$.

in which the right hand side square is a pullback, and the left hand side is defined by factoring the induced map on kernel relations $R[f'] \rightarrow R[f]$ through the direct image $S = l(R[f'])$ under l . Since the right square is a pullback, the left square represents a cartesian split epimorphism of equivalence relations. The factorization of this cartesian morphism induced by l yields two cartesian maps of equivalence relations, cf. Lemma 1.23. Note that the second $(X'', S) \rightarrow (X, R[f])$ is a cartesian *split* epimorphism. Efficient regularity implies then that the equivalence relation S is effective with quotient $f'' : U'' \twoheadrightarrow V''$, defining a point (ρ, σ) over Y .

This induces (by a well-known result of Barr and Kock, cf. [5, Lemma A.5.8]) a decomposition of the right pullback into two pullbacks. The induced regular epimorphism $\bar{l} : (r, s) \twoheadrightarrow (\rho, \sigma)$ has the required properties, namely $f^*(\bar{l}) = l$. \square

Proposition 3.12. *In an efficiently regular Mal'cev category with binary sums, a regular epimorphism $f : X \twoheadrightarrow Y$ is an affine extension if and only if for each object Z either of the following two diagrams*

$$\begin{array}{ccc} X + Z & \xrightarrow{f+Z} & Y + Z \\ \pi_X^Z \downarrow & & \downarrow \pi_Y^Z \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X + Z & \xrightarrow{f+Z} & Y + Z \\ \theta_{X,Z} \downarrow & & \downarrow \theta_{Y,Z} \\ X \times Z & \xrightarrow{f \times Z} & Y \times Z \end{array}$$

is a downward-oriented pullback square.

Proof. If f is an affine extension then the downward-oriented left square is a pullback because the upward-oriented left square is a pushout, cf. Section 3.2. Moreover, the outer rectangle of the following diagram

$$\begin{array}{ccc} X + Z & \xrightarrow{f+Z} & Y + Z \\ \theta_{X,Z} \downarrow & & \downarrow \theta_{Y,Z} \\ X \times Z & \xrightarrow{f \times Z} & Y \times Z \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback if and only if the upper square is a pullback, because the lower square is always a pullback.

Assume conversely that the downward oriented left square is a pullback. In a regular Mal'cev category, pointed base-change along a regular epimorphism is fully faithful so that f is affine whenever f^* is essentially surjective. Lemma 3.11 shows that in an efficiently regular Mal'cev category f^* is saturated on quotients. It suffices thus to show that in the fibre over X each point is the quotient of a point for which the unit of the pointed base-change adjunction is invertible.

Since for each object Z , the undotted downward-oriented square

$$\begin{array}{ccc}
X + Z & \xrightarrow{f+Z} & Y + Z \\
\pi_X^Z \downarrow & \langle 1_X, r \rangle & \downarrow \pi_Y^Z \\
X & \xrightarrow{f} & Y
\end{array}$$

is a pullback, the dotted downward-oriented square (which is induced by an arbitrary morphism $r : Z \rightarrow X$) is a pullback as well. This holds in any regular Mal'cev category, since the whole diagram represents a natural transformation of reflexive graphs, cf. [7]. It follows that the point $(\langle 1_X, r \rangle, \iota_X^Z) : X + Z \rightrightarrows X$ has an invertible unit with respect to the pointed base-change adjunction $(f_!, f^*)$.

Now, an arbitrary point $(r, s) : Z \rightrightarrows X$ can be realized as a quotient

$$\begin{array}{ccc}
Z & \xleftarrow{\langle s, 1_Z \rangle} & X + Z \\
r \swarrow & \iota_X^Z \nearrow & \\
& X & \swarrow \langle 1_X, r \rangle
\end{array}$$

of the latter point $(\langle 1_X, r \rangle, \iota_X^Z) : X + Y \rightrightarrows X$ with invertible unit. \square

We end this section with several properties of affine extensions in semi-abelian categories. They will only be used in Section 6.

Proposition 3.13. *In a semi-abelian category, a regular epimorphism $f : X \twoheadrightarrow Y$ is an affine extension if and only if either of the following conditions is satisfied:*

- (a) *for each object Z , the induced map $f \diamond Z : X \diamond Z \rightarrow Y \diamond Z$ is invertible, where $X \diamond Z$ stands for the kernel of $\theta_{X,Z} : X + Z \twoheadrightarrow X \times Z$;*
- (b) *every pushout of f along a split monomorphism is a pullback.*

Proof. That condition (a) characterizes affine extensions follows from Proposition 3.12 and protomodularity. The necessity of condition (b) follows from Section 3.2. The sufficiency of condition (b) follows from the “pullback cancellation property” in semi-abelian categories, cf. [5, Proposition 4.1.4]. \square

Remark 3.14. This product $X \diamond Z$ is often called the *co-smash product*, since it is the precise dual of the *smash product* for pointed objects, as investigated by Carboni-Janelidze [17] in the context of lextensive categories. The co-smash product $X \diamond Z$ coincides in semi-abelian categories with the second cross-effect $cr_2(X, Z)$ of the identity functor, cf. Definition 5.1 and [38, 39]. Since the co-smash product is in general not associative (cf. [17]), parentheses should be used with care.

Proposition 3.15. *Let $Y \twoheadrightarrow W \leftarrow Z$ and $\bar{Y} \twoheadrightarrow \bar{W} \leftarrow \bar{Z}$ be cospans in the fibre $\text{Pt}_X(\mathbb{D})$ of a semi-abelian category \mathbb{D} . Let $f : Y \twoheadrightarrow \bar{Y}$, $g : Z \twoheadrightarrow \bar{Z}$, $h : W \twoheadrightarrow \bar{W}$ be affine extensions in \mathbb{D} inducing a map of cospans in $\text{Pt}_X(\mathbb{D})$. Assume furthermore that the first cospan realizes W as the binary sum of Y and Z in $\text{Pt}_X(\mathbb{D})$.*

Then the second cospan realizes \bar{W} as the binary sum of \bar{Y} and \bar{Z} in $\text{Pt}_X(\mathbb{D})$ if and only if the kernel cospan $K[f] \twoheadrightarrow K[h] \leftarrow K[g]$ is strongly epimorphic in \mathbb{D} .

Proof. Let us consider the following commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i} & Y & \xrightarrow{f} & \bar{Y} \\
 j \downarrow & & \downarrow & & \downarrow \\
 Z & \xrightarrow{\quad} & W & \xrightarrow{h_1} & W_1 \\
 g \downarrow & & \downarrow h_2 & \searrow h & \downarrow \\
 \bar{Z} & \xrightarrow{\quad} & W_2 & \xrightarrow{\quad} & \bar{W}
 \end{array}$$

in which i (resp. j) denotes the section of the point Y (resp. Z) over X , and all little squares except the lower right one are pushouts. It follows that the outer square is a pushout (i.e. $\bar{W} = \bar{Y} +_X \bar{Z}$) if and only if the lower right square is a pushout. According to Carboni-Kelly-Pedicchio [18, Theorem 5.2] this happens if and only if the kernel relation $R[h]$ is the join of the kernel relations $R[h_1]$ and $R[h_2]$. In a semi-abelian category this is the case if and only if the kernel $K[h]$ is generated as normal subobject of W by the kernels $K[h_1]$ and $K[h_2]$, resp. (since h_1 and h_2 are affine extensions) by the kernels $K[f]$ and $K[g]$, cf. Proposition 3.13b.

Now, h is also an affine extension so that by Proposition 3.4, the kernel $K[h]$ is a central subobject of W . In particular, any subobject of $K[h]$ is central and normal in W (cf. the characterization of normal subobjects in semi-abelian categories by Mantovani-Metere [54, Theorem 6.3]). Therefore, generating $K[h]$ as normal subobject of W amounts to the same as generating $K[h]$ as subobject of W . \square

4. ASPECTS OF NILPOTENCY

Recall that a morphism is called n -fold centrally decomposable if it is the composite of n morphisms with central kernel relation.

For consistency, a monomorphism is called 0-fold centrally decomposable, and an isomorphism a 0-fold central extension.

Proposition 4.1. *For all objects X, Y of a σ -pointed n -nilpotent Mal'cev category, the comparison map $\theta_{X,Y} : X + Y \rightarrow X \times Y$ is $(n-1)$ -fold centrally decomposable.*

Proof. In a pointed n -nilpotent Mal'cev category, each object maps to an abelian group object through an $(n-1)$ -fold centrally decomposable morphism. Therefore, there is such a morphism $\phi_{X,Y} : X + Y \rightarrow A$. Since A is an abelian group object, the restrictions of $\phi_{X,Y}$ to the two summands commute and $\phi_{X,Y}$ factors

$$\begin{array}{ccc}
 X + Y & \xrightarrow{\theta_{X,Y}} & X \times Y \\
 \phi_{X,Y} \searrow & & \swarrow \psi_{X,Y} \\
 & A &
 \end{array}$$

so that $\theta_{X,Y}$ is $(n-1)$ -fold centrally decomposable by Lemma 1.5. \square

Proposition 4.2. *For any finitely cocomplete regular pointed Mal'cev category, the following pushout square*

$$\begin{array}{ccc}
 X + X & \xrightarrow{\theta_{X,X}} & X \times X \\
 \langle 1_X, 1_X \rangle \downarrow & & \downarrow \\
 X & \xrightarrow{\quad} & A(X)
 \end{array}$$

defines the abelianization $A(X)$ of X . In particular, the lower row can be identified with the unit $\eta_X^1 : X \rightarrow I^1(X)$ of the epireflection of Theorem 2.14.

Proof. The first assertion follows by combining [5, Proposition 1.7.5, Theorems 1.9.5 and 1.9.11] with the fact that pointed Mal'cev categories are strongly unital in the sense of the second author, cf. [5, Corollary 2.2.10]. The second assertion expresses the fact that $X \twoheadrightarrow A(X)$ and $X \twoheadrightarrow I^1(X)$ share the same universal property. \square

Theorem 4.3. *A σ -pointed exact Mal'cev category is n -nilpotent if and only if for all objects X, Y the comparison map $\theta_{X,Y} : X + Y \rightarrow X \times Y$ is an $(n-1)$ -fold central extension.*

Proof. Proposition 4.1 shows the necessity of the condition. For its sufficiency, consider the pushout square of Proposition 4.2, which is regular by Corollary 1.11. Therefore, the unit $\eta_X^1 : X \rightarrow I^1(X)$ of the first Birkhoff reflection is an $(n-1)$ -fold central extension by Proposition 1.16, and all objects are n -nilpotent. \square

Corollary 4.4. *For a σ -pointed exact Mal'cev category, the following three properties are equivalent:*

- (a) *the category is 1-nilpotent;*
- (b) *the category is linear;*
- (c) *the category is abelian.*

Proof. The equivalence of (a) and (b) follows from Theorem 4.3. The equivalence of (b) and (c) follows from the fact that a σ -pointed Mal'cev category is additive if and only if it is linear, cf. [5, Theorem 1.10.14], together with the well-known fact (due to Miles Tierney) that abelian categories are precisely the additive categories among exact categories. \square

Theorem 4.5. *For a σ -pointed exact Mal'cev category, the following five properties are equivalent:*

- (a) *all objects are 2-nilpotent;*
- (b) *for all X , abelianization $\eta_X^1 : X \rightarrow I^1(X)$ is a central extension;*
- (b') *for all X , abelianization $\eta_X^1 : X \rightarrow I^1(X)$ is an affine extension;*
- (c) *for all X, Y , the map $\theta_{X,Y} : X + Y \rightarrow X \times Y$ is a central extension;*
- (c') *for all X, Y , the map $\theta_{X,Y} : X + Y \rightarrow X \times Y$ is an affine extension.*

Proof. Properties (a) and (b) are equivalent by definition of 2-nilpotency. Theorem 4.3 shows that (b) and (c) are equivalent. Theorem 3.5 and Proposition 3.4 imply that (b) and (b') are equivalent.

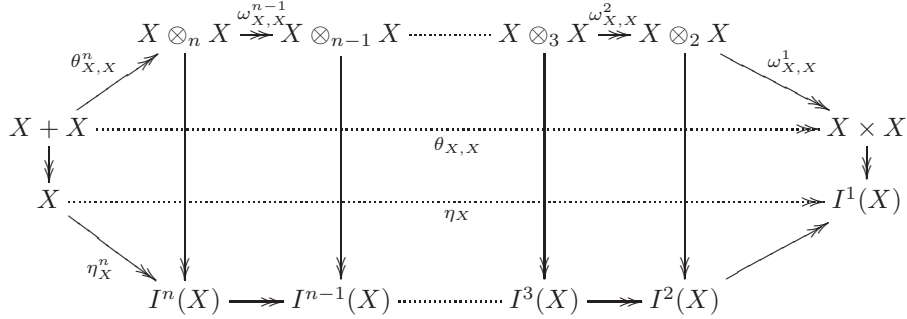
Finally, since the first Birkhoff reflection I^1 preserves binary sums and binary products (cf. Proposition 2.10), we get $I^1(\theta_{X,Y}) = \theta_{I^1(X), I^1(Y)}$ which is invertible in the subcategory of 1-nilpotent objects by Proposition 4.1. It follows that under assumption (a), the map $\theta_{X,Y}$ is an affine extension by Theorem 3.5, which is property (c'). Conversely, (c') implies (c) by Proposition 3.4. \square

4.6. Niltensor products. In order to extend Proposition 4.5 to higher n we introduce here a new family of binary products, called *niltensor products*.

For any finitely cocomplete pointed regular Mal'cev category $(\mathbb{D}, \star_{\mathbb{D}})$ the n -th *niltensor product* $X \otimes_n Y$ is defined by factorizing the comparison map $\theta_{X,Y}$ universally into a regular epimorphism $\theta_{X,Y}^n : X + Y \rightarrow X \otimes_n Y$ followed by an

$$\begin{array}{c}
X \otimes_n Y \xrightarrow{\omega_{X,Y}^{n-1}} X \otimes_{n-1} Y \cdots \cdots \cdots X \otimes_3 Y \xrightarrow{\omega_{X,Y}^2} X \otimes_2 Y \\
\theta_{X,Y}^n \nearrow \quad \theta_{X,Y}^{n-1} \nearrow \quad \theta_{X,Y}^2 \nearrow \quad \omega_{X,Y}^1 \searrow \\
X + Y \xrightarrow{\theta_{X,Y}} X \times Y
\end{array}$$

Proposition 4.7. *In a σ -pointed exact Mal'cev category, the following diagram is an iterated pushout diagram*



Proof. This follows from Corollary 1.11 and Propositions 1.16 and 4.2. \square

(a) *all objects are n -nilpotent;*
(b) *for all X , the $(n-1)$ th unit $\eta_X^{n-1}: X \twoheadrightarrow I^{n-1}(X)$ is a central extension;*
(b') *for all X , the $(n-1)$ th unit $\eta_X^{n-1}: X \twoheadrightarrow I^{n-1}(X)$ is an affine extension;*
(c) *for all X, Y , the map $\theta_{X,Y}^{n-1}: X + Y \rightarrow X \otimes_{n-1} Y$ is a central extension;*
(c') *for all X, Y , the map $\theta_{X,Y}^{n-1}: X + Y \rightarrow X \otimes_{n-1} Y$ is an affine extension.*

Finally, Proposition 1.16 implies that the Birkhoff reflection I^{n-1} takes the comparison map $\theta_{X,Y}^{n-1}$ to the corresponding map $\theta_{I^{n-1}(X), I^{n-1}(Y)}^{n-1}$ for the $(n-1)$ -nilpotent objects $I^{n-1}(X)$ and $I^{n-1}(Y)$. Since the $(n-1)$ -nilpotent objects form an $(n-1)$ -nilpotent Birkhoff subcategory, Theorem 4.3 shows that the latter map must be invertible; therefore, (a) implies (c') by Theorem 3.5. Conversely, (c') implies (c) by Proposition 3.4. \square

Definition 4.9. A σ -pointed Mal'cev category is said to be pseudo-additive (resp. pseudo- n -additive) if for all X, Y , the map $\theta_{X,Y} : X + Y \rightarrow X \times Y$ (resp. $\theta_{X,Y}^n : X + Y \rightarrow X \otimes_n Y$) is an affine extension.

Proposition 4.10. *A σ -pointed exact Mal'cev category is pseudo-additive (i.e. 2-nilpotent) if and only if the following diagram*

$$\begin{array}{ccc} (X + Y) + Z & \xrightarrow{\theta_{X,Y} + Z} & (X \times Y) + Z \\ \theta_{X+Y,Z} \downarrow & & \downarrow \theta_{X \times Y, Z} \\ (X + Y) \times Z & \xrightarrow{\theta_{X,Y} \times Z} & (X \times Y) \times Z \end{array}$$

is a pullback for all objects X, Y, Z .

Proof. This follows from Theorem 4.5 and Proposition 3.12. \square

Proposition 4.11. *A σ -pointed exact Mal'cev category is pseudo- $(n-1)$ -additive (i.e. n -nilpotent) if and only if the following diagram*

$$\begin{array}{ccc} (X + Y) + Z & \xrightarrow{\theta_{X+Y,Z}} & (X + Y) \times Z \\ \theta_{X,Y}^{n-1} + Z \downarrow & & \downarrow \theta_{X,Y}^{n-1} \times Z \\ (X \otimes_{n-1} Y) + Z & \xrightarrow{\theta_{X \otimes_{n-1} Y, Z}} & (X \otimes_{n-1} Y) \times Z \end{array}$$

is a pullback for all objects X, Y, Z .

Proof. This follows from Theorem 4.8 and Proposition 3.12. \square

We end this section with a general remark about the behaviour of n -nilpotency under slicing and passage to the fibres. Note that any left exact functor between Mal'cev categories preserves central equivalence relations, morphisms with central kernel relation, and consequently n -nilpotent objects.

Proposition 4.12. *If \mathbb{D} is an n -nilpotent Mal'cev category, then so are any of its slice categories \mathbb{D}/Y and of its fibres $\text{Pt}_Y(\mathbb{D})$.*

Proof. The slices \mathbb{D}/Y of a Mal'cev category \mathbb{D} are again Mal'cev categories. Moreover, base-change $\omega_Y^* : \mathbb{D} \rightarrow \mathbb{D}/Y$ is a left exact functor so that the objects of \mathbb{D}/Y of the form $\omega_Y^*(X) = p_Y : Y \times X \rightarrow Y$ are n -nilpotent provided \mathbb{D} is an n -nilpotent Mal'cev category. We can conclude with Proposition 2.3 by observing that *any* object $f : X \rightarrow Y$ of \mathbb{D}/Y may be considered as a subobject

$$\begin{array}{ccc} X & \xrightarrow{(f, 1_X)} & Y \times X \\ & \searrow f & \swarrow p_Y \\ & Y & \end{array}$$

of $\omega_Y^*(X)$ in \mathbb{D}/Y . The proof for the fibres is the same as for the slices, since any object (r, s) of $\text{Pt}_Y(\mathbb{D})$ may be considered as a subobject

$$\begin{array}{ccc} X & \xrightarrow{(r, 1_X)} & Y \times X \\ \swarrow s & & \swarrow p_Y \\ & Y & \searrow (1_Y, s) \\ & \nwarrow r & \end{array}$$

of the projection $p_Y : Y \times X \rightarrow Y$ splitted by $(1_Y, s) : Y \rightarrow Y \times X$. \square

5. QUADRATIC IDENTITY FUNCTORS

We have seen that 1-nilpotency has much to do with linear identity functors (cf. Corollary 4.4). We now investigate the relationship between 2-nilpotency and quadratic identity functors, and below in Section 6, the relationship between n -nilpotency and identity functors of degree n . While a linear functor takes binary sums to binary products, a quadratic functor takes certain cubes constructed out of triple sums to limit cubes. This is the beginning of a whole hierarchy assigning degree $\leq n$ to a functor whenever the functor takes certain $(n + 1)$ -dimensional cubes constructed out of iterated sums to limit cubes.

This definition of *degree* of a functor is much inspired by Goodwillie [29] who described polynomial approximations of a homotopy functor in terms of their behaviour on certain cubical diagrams. Eilenberg-Mac Lane [24] defined the degree of a functor with values in an *abelian* category by a vanishing condition of so-called *cross-effects*. Our definition of degree does not need cross-effects. Yet, a functor with values in a *semi-abelian* (or homological [5]) category is of degree $\leq n$ precisely when all its cross-effects of order $n + 1$ vanish, cf. Corollary 6.18. Our cubical cross-effects coincide with those of Hartl-Loiseau [38] and Hartl-Van der Linden [39] which are defined as kernel intersections.

There are several other places in literature where degree n functors, and especially quadratic functors, are studied in a non-additive context, most notably Baues-Pirashvili [1], Johnson-McCarthy [49] and Hartl-Vespa [40]. It is however significative that in all these places, the definition of a degree n functor is based on a vanishing condition of cross-effects, completely bypassing Goodwillie's cubical approach. The cubical approach has its advantages, even in our algebraic setting. It simplifies the proofs and yields a characterization of degree n functors in terms of n -folded objects, cf. Section 6 below.

An n -cube in a category \mathbb{E} is given by a functor $\Xi : [0, 1]^n \rightarrow \mathbb{E}$ with domain the n -fold cartesian product $[0, 1]^n$ of the arrow category $[0, 1]$.

The category $[0, 1]$ has two objects 0, 1 and exactly one non-identity arrow $0 \rightarrow 1$. Thus, an n -cube in \mathbb{E} is given by objects $\Xi(\epsilon_1, \dots, \epsilon_n)$ in \mathbb{E} with $\epsilon_i \in \{0, 1\}$, and arrows $\xi_{\epsilon_1, \dots, \epsilon_n}^{\epsilon'_1, \dots, \epsilon'_n} : \Xi(\epsilon_1, \dots, \epsilon_n) \rightarrow \Xi(\epsilon'_1, \dots, \epsilon'_n)$ in \mathbb{E} , one for each arrow in $[0, 1]^n$, which compose in an obvious way.

To each n -cube Ξ we associate a *punctured* n -cube $\check{\Xi}$ obtained by restriction of Ξ to the full subcategory of $[0, 1]^n$ spanned by the objects $(\epsilon_1, \dots, \epsilon_n) \neq (0, \dots, 0)$.

Definition 5.1. Let $(\mathbb{E}, \star_{\mathbb{E}})$ be a σ -pointed category. For each n -tuple of objects (X_1, \dots, X_n) of \mathbb{E} we denote by Ξ_{X_1, \dots, X_n} the following n -cube:

- $\Xi_{X_1, \dots, X_n}(\epsilon_1, \dots, \epsilon_n) = X_1(\epsilon_1) + \dots + X_n(\epsilon_n)$
with $X(0) = X$ and $X(1) = \star_{\mathbb{E}}$;
- $\xi_{\epsilon_1, \dots, \epsilon_n}^{\epsilon'_1, \dots, \epsilon'_n} = j_{\epsilon_1}^{\epsilon'_1} + \dots + j_{\epsilon_n}^{\epsilon'_n}$
where $j_{\epsilon}^{\epsilon'}$ is the identity if $\epsilon = \epsilon'$, resp. the null morphism if $\epsilon \neq \epsilon'$.

A functor of σ -pointed categories $F : (\mathbb{E}, \star_{\mathbb{E}}) \rightarrow (\mathbb{E}', \star_{\mathbb{E}'})$ is called of degree $\leq n$ if

- $F(\star_{\mathbb{E}}) \cong \star_{\mathbb{E}'}$;
- for each $(n + 1)$ -cube $\Xi_{X_1, \dots, X_{n+1}}$ in \mathbb{E} , the image-cube $F(\Xi_{X_1, \dots, X_{n+1}})$ is a limit-cube in \mathbb{E}' , i.e. $F(X_1 + \dots + X_{n+1})$ may be identified with the limit of the punctured image-cube $F(\check{\Xi}_{X_1, \dots, X_{n+1}})$ in \mathbb{E}' .

A functor of degree ≤ 1 (resp. ≤ 2) is called linear (resp. quadratic).

A σ -pointed category is called *linear* (resp. *quadratic*) if its identity functor is. If \mathbb{E}' has pullbacks, the limit over the punctured image-cube $F(\check{\Xi}_{X_1, \dots, X_{n+1}})$ is denoted

$$P_{X_1, \dots, X_{n+1}}^F = \varprojlim_{\check{\Xi}_{X_1, \dots, X_{n+1}}} F|_{\check{\Xi}_{X_1, \dots, X_{n+1}}}$$

and the associated comparison map is denoted

$$\theta_{X_1, \dots, X_{n+1}}^F : F(X_1 + \dots + X_{n+1}) \rightarrow P_{X_1, \dots, X_{n+1}}^F.$$

The $(n+1)$ -st cross-effect of the functor $F : \mathbb{E} \rightarrow \mathbb{E}'$ is the total kernel of the image-cube $F(\Xi_{X_1, \dots, X_{n+1}})$, i.e. the kernel of the comparison map $\theta_{X_1, \dots, X_{n+1}}^F$:

$$cr_{n+1}^F(X_1, \dots, X_{n+1}) = K[\theta_{X_1, \dots, X_{n+1}}^F].$$

If F is the identity functor, the symbol F will be dropped from the notation. A functor $F : \mathbb{E} \rightarrow \mathbb{E}'$ has degree $\leq n$ if and only if for all $(n+1)$ -tuples (X_1, \dots, X_{n+1}) of objects of \mathbb{E} , the comparison map $\theta_{X_1, \dots, X_{n+1}}^F$ is invertible.

Our *total-kernel* definition of the cross-effect $cr_{n+1}^F(X_1, \dots, X_{n+1})$ (inspired by Goodwillie [29, pg. 676]) yields the same subobject of $F(X_1 + \dots + X_{n+1})$ as the *kernel-intersection* definition of Hartl-Loiseau [38] and Hartl-Van der Linden [39]. The latter is the precise dual of the $(n+1)$ -fold smash product of Carboni-Janelidze [17], cf. Remark 3.14 and also Remark 6.2 where this duality between cross-effect and smash-product is discussed in more detail.

Indeed, each of the $n+1$ “contraction morphisms”

$$\pi_{\widehat{X}_i}^F : F(X_1 + \dots + X_{n+1}) \rightarrow F(X_1 + \dots + \widehat{X}_i + \dots + X_{n+1}), \quad 1 \leq i \leq n+1,$$

factors through $\theta_{X_1, \dots, X_{n+1}}^F$ so that we get a composite morphism

$$r_F : F(X_1 + \dots + X_{n+1}) \xrightarrow{\theta_{X_1, \dots, X_{n+1}}^F} P_{X_1, \dots, X_{n+1}}^F \hookrightarrow \prod_{i=1}^{n+1} F(X_1 + \dots + \widehat{X}_i + \dots + X_{n+1})$$

embedding the limit construction $P_{X_1, \dots, X_{n+1}}^F$ into an $(n+1)$ -fold cartesian product. Therefore, the kernel of $\theta_{X_1, \dots, X_{n+1}}^F$ coincides with the kernel of r_F

$$K[r_F] = \bigcap_{i=1}^{n+1} K[\pi_{\widehat{X}_i}^F],$$

which is precisely the kernel-intersection of [38, 39] serving as their definition for the $(n+1)$ st cross-effect $cr_{n+1}^F(X_1, \dots, X_{n+1})$ of the functor F .

For $n=1$ and $F = id_{\mathbb{E}}$ we get the following 2-cube $\Xi_{X,Y}$

$$\begin{array}{ccc} X + Y & \xrightarrow{\pi_{\widehat{X}}} & Y \\ \pi_{\widehat{Y}} \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & \star_{\mathbb{E}} \end{array}$$

so that the limit $P_{X,Y}$ of the punctured 2-cube is $X \times Y$ and the comparison map

$$\theta_{X,Y} : X + Y \rightarrow X \times Y$$

is the one already used before. In particular, the just introduced notion of *linear category* is the usual one.

For $n=2$ and $F = id_{\mathbb{E}}$ we get the following 3-cube $\Xi_{X,Y,Z}$

$$\begin{array}{ccccc}
& & Y+Z & \xrightarrow{\pi_Y} & Z \\
& \nearrow \pi_X & \uparrow & \nearrow \pi_X & \uparrow \\
X+Y+Z & \xrightarrow{\pi_Y} & X+Z & & \\
& \searrow \pi_Z & \downarrow \pi_Z & \searrow \pi_Z & \downarrow \pi_Z \\
& \nearrow \pi_X & Y & \xrightarrow{\pi_Y} & *_{\mathbb{E}} \\
X+Y & \xrightarrow{\pi_Y} & X & \nearrow \pi_Z & \\
& & & \nearrow \pi_Z &
\end{array}$$

which induces a split natural transformation of 2-cubes:

$$\Xi_{X,Y} + Z \rightleftharpoons \Xi_{X,Y}$$

For sake of simplicity, we denote by $+Z$ the functor $\mathbb{E} \rightarrow \text{Pt}_Z(\mathbb{E})$ which takes an object X to $X + Z \rightarrow Z$ with obvious section, and similarly, we denote by $\times Z$ the functor $\mathbb{E} \rightarrow \text{Pt}_Z(\mathbb{E})$, which takes an object X to $X \times Z \rightarrow Z$ with obvious section.

The previous split natural transformation of 2-cubes induces a natural transformation of split epimorphisms

$$\begin{array}{ccc}
X+Y+Z & \xrightarrow{\theta_{X,Y}^{+Z}} & (X+Z) \times_Z (Y+Z) \\
\pi_Z \updownarrow & \textcircled{2} & \updownarrow \pi_Z \\
X+Y & \xrightarrow{\theta_{X,Y}} & X \times Y
\end{array}$$

the comparison map of which may be identified with the comparison map

$$\theta_{X,Y,Z} : X+Y+Z \rightarrow P_{X,Y,Z}$$

of $\Xi_{X,Y,Z}$. In particular, the category \mathbb{E} is quadratic if and only if square (2) is a pullback square. Notice that in a regular Mal'cev category, the downward-oriented square (2) is necessarily a regular pushout by Corollary 1.10.

Proposition 5.2. *In a σ -pointed regular Mal'cev category, the comparison map $\theta_{X,Y,Z}$ is a regular epimorphism with kernel relation $R[\pi_X] \cap R[\pi_Y] \cap R[\pi_Z]$.*

Proof. The first assertion expresses the regularity of pushout (2), the second follows from identities $R[\theta_{X,Y}^{+Z}] = R[\pi_X] \cap R[\pi_Y]$ and $R[\theta_{X,Y,Z}] = R[\theta_{X,Y}^{+Z}] \cap R[\pi_Z]$ which hold because both, $\theta_{X,Y}^{+Z}$ and $\theta_{X,Y,Z}$, are comparison maps in regular pushouts. \square

Lemma 5.3. *A σ -pointed category with pullbacks is quadratic if and only if square (2') of the following diagram*

$$\begin{array}{ccccc}
X+Y+Z & \xrightarrow{\theta_{X+Y,Z}} & (X+Y) \times Z & \xrightarrow{\quad} & X+Y \\
\theta_{X,Y}^{+Z} \downarrow & & \downarrow \theta_{X,Y}^{\times Z} & \textcircled{2''} & \downarrow \theta_{X,Y} \\
(X+Z) \times_Z (Y+Z) & \xrightarrow{\theta_{X,Z} \times_Z \theta_{Y,Z}} & (X \times Z) \times_Z (Y \times Z) & \xrightarrow{\quad} & X \times Y
\end{array}$$

(2')

is a pullback square.

Proof. Composing squares (2') and (2'') yields square (2) above. Square (2'') is a pullback since $(X \times Z) \times_Z (Y \times Z)$ is canonically isomorphic to $(X \times Y) \times Z$. \square

5.4. The main diagram. We shall now give several criteria for quadraticity. For this we consider the following diagram

$$\begin{array}{ccccc}
 (X + Y) \diamond Z & \xrightarrow{\quad} & (X + Y) + Z & \xrightarrow{\theta_{X+Y,Z}} & (X + Y) \times Z \\
 \theta_{X,Y} \diamond Z \downarrow & & \downarrow \theta_{X,Y} + Z & \textcircled{a} & \downarrow \theta_{X,Y} \times Z \\
 (X \times Y) \diamond Z & \xrightarrow{\quad} & (X \times Y) + Z & \xrightarrow{\theta_{X \times Y,Z}} & (X \times Y) \times Z \\
 \varphi_{X,Y}^Z \downarrow & & \downarrow \phi_{X,Y}^Z & \textcircled{b} & \downarrow \mu_{X,Y}^Z \\
 (X \diamond Z) \times (Y \diamond Z) & \xrightarrow{\quad} & (X + Z) \times_Z (Y + Z) & \xrightarrow{\theta_{X,Z} \times_Z \theta_{Y,Z}} & (X \times Z) \times_Z (Y \times Z)
 \end{array}$$

in which the vertical composite morphisms from left to right are $\theta_{X,Y}^{\diamond Z}, \theta_{X,Y}^{+Z}, \theta_{X,Y}^{\times Z}$, the horizontal morphisms on the left are the kernel-inclusions of the horizontal regular epimorphisms on their right, and $\mu_{X,Y}^Z$ is the canonical isomorphism.

Observe that square (b) is a pullback if and only if the canonical map

$$\phi_{X,Y}^Z : (X \times Y) + Z \rightarrow (X + Z) \times_Z (Y + Z)$$

is invertible. This is the case if and only if for each Z pointed cobase-change

$$(\alpha_Z)_! : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D})$$

along the initial map $\alpha_Z : \star_{\mathbb{D}} \rightarrow Z$ preserves binary products, cf. Section 3.2.

Proposition 5.5. *A σ -pointed regular Mal'cev category is quadratic if and only if squares (a) and (b) of the main diagram are pullback squares.*

Proof. Since composing squares (a) and (b) yields square (2') of Lemma 5.3, the condition is sufficient. Lemma 1.23 and Corollary 1.20 imply that the condition is necessary as well. \square

Theorem 5.6. *A σ -pointed exact Mal'cev category is quadratic if and only if it is 2-nilpotent and pointed cobase-change along initial maps preserves binary products.*

Proof. By Proposition 5.5, the category is quadratic if and only if the squares (a) and (b) are pullback squares, i.e. the category is 2-nilpotent by Proposition 4.10, and pointed cobase-change along initial maps preserves binary products. \square

Corollary 5.7. *A semi-abelian category is quadratic if and only if either of the following three equivalent conditions is satisfied:*

- (a) *all objects are 2-nilpotent, and the comparison maps*

$$\varphi_{X,Y}^Z : (X \times Y) \diamond Z \rightarrow (X \diamond Z) \times (Y \diamond Z)$$

are invertible for all objects X, Y, Z ;

- (b) *the third cross-effects of the identity functor $cr_3(X, Y, Z) = K[\theta_{X,Y,Z}]$ vanish for all objects X, Y, Z ;*

- (c) *the co-smash product is linear, i.e. the canonical comparison maps*

$$\theta_{X,Y}^{\diamond Z} : (X + Y) \diamond Z \rightarrow (X \diamond Z) \times (Y \diamond Z)$$

are invertible for all objects X, Y, Z .

Proof. Theorem 5.6 shows that condition (a) amounts to quadraticity.

For condition (b) note that by protomodularity the cross-effect $K[\theta_{X,Y,Z}]$ vanishes if and only if the regular epimorphism $\theta_{X,Y,Z}$ is invertible.

The equivalence of conditions (b) and (c) follows from the isomorphism of kernels $K[\theta_{X,Y,Z}] \cong K[\theta_{X,Y}^{\circ Z}]$. The latter is a consequence of the 3×3 -lemma which, applied to main diagram 5.4 and square (2), yields the chain of isomorphisms

$$K[\theta_{X,Y}^{\circ Z}] \cong K[K[\theta_{X,Y}^{+Z}] \rightarrow K[\theta_{X,Y}^{\times Z}]] \cong K[\theta_{X,Y,Z}].$$

□

5.8. Algebraic distributivity and algebraic extensivity. –

We shall see that in a pseudo-additive regular Mal'cev category $(\mathbb{D}, \star_{\mathbb{D}})$, pointed cobase-change along initial maps $\alpha_Z : \star_{\mathbb{D}} \rightarrow Z$ preserves binary products if and only if pointed base-change along terminal maps $\omega_Z : Z \rightarrow \star_{\mathbb{D}}$ preserves binary sums. The latter condition means that for all objects X, Y, Z , the following square

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & Y \times Z \\ \downarrow & & \downarrow \\ X \times Z & \xrightarrow{\quad} & (X + Y) \times Z \end{array}$$

is a pushout, inducing thus for all objects X, Y, Z an isomorphism

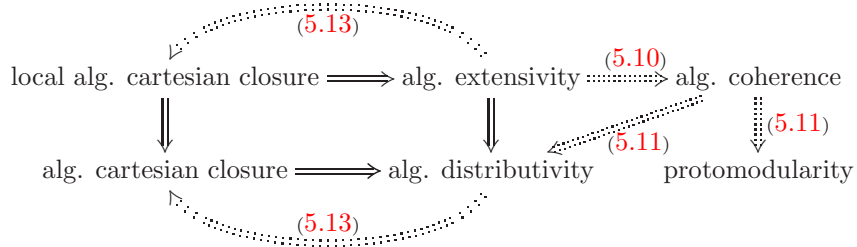
$$(X \times Z) +_Z (Y \times Z) \cong (X + Y) \times Z$$

which can be considered as an algebraic distributivity law. This suggests the following definitions, where the added adjective “algebraic” means here that the familiar definition has to be modified by replacing the *slices* of the category with the *fibres* of the fibration of points, cf. Section 3.2 and Carboni-Lack-Walters [19].

Definition 5.9. *A category with pullbacks of split epimorphisms is algebraically distributive if pointed base-change along terminal maps preserves binary sums.*

A category with pullbacks of split epimorphisms and pushouts of split monomorphisms is algebraically extensive if any pointed base-change preserves binary sums.

We get the following implications between several in literature studied “algebraic” notions, where we assume that pullbacks of split epimorphisms and (when needed) pushouts of split monomorphisms exist:



The existence of centralizers implies *algebraic cartesian closure* [14] and hence *algebraic distributivity*, cf. Section 1.21. The categories of groups and of Lie algebras are not only algebraically cartesian closed, but also *locally algebraically cartesian closed* [35, 36], which means that any pointed base-change admits a right adjoint. *Algebraic coherence*, cf. Cigoli-Gray-Van der Linden [21], requires any pointed base-change to be *coherent*, i.e. to preserve strongly epimorphic cospans.

Lemma 5.10. *An algebraically extensive regular category is algebraically coherent.*

Proof. In a regular category, pointed base-change preserves regular epimorphisms. Henceforth, if the fibres have binary sums and pointed base-change preserves them, pointed base-change also preserves strongly epimorphic cospans. \square

Lemma 5.11 (cf. [11], Theorem 3.10 and [21], Theorem 6.1). *An algebraically coherent pointed Mal'cev category is protomodular and algebraically distributive.*

Proof. To any split epimorphism $(r, s) : Y \rightrightarrows X$ we associate the split epimorphism $(\bar{r} = r \times 1_X, \bar{s} = s \times 1_X) : Y \times X \rightrightarrows X \times X$

$$\begin{array}{ccc}
 Y \times X & \xrightleftharpoons[(\bar{s})]{(\bar{r})} & X \times X \\
 \swarrow (s, 1_X) \quad \searrow (1_X, 1_X) & & \\
 & X & \\
 \nwarrow p_2 \quad \nearrow p_2 & &
 \end{array}$$

in the fibre over X . The kernel of (\bar{r}, \bar{s}) may be identified with the given point (r, s) over X where the kernel-inclusion is defined by $(1_Y, r) : Y \rightarrow Y \times X$. Kernel-inclusion and section strongly generate the point $Y \times X$ over X , cf. [11, Proposition 3.7]. Pointed base-change along $\alpha_X : \star \rightarrow X$ takes (\bar{r}, \bar{s}) back to (r, s) , so that by algebraic coherence, section and kernel-inclusion of (r, s) strongly generate Y . In a pointed category this amounts to protomodularity.

For the second assertion observe that if F and G are composable coherent functors such that G is conservative and GF preserves binary sums, then F preserves binary sums as well; indeed, the isomorphism $GF(X) + GF(Y) \rightarrow GF(X + Y)$ decomposes into two isomorphisms $GF(X) + GF(Y) \rightarrow G(F(X) + F(Y)) \rightarrow GF(X + Y)$. This applies to $F = \omega_Z^*$ and $G = \alpha_Z^*$ (where $\alpha_Z : \star \rightarrow Z$ and $\omega_Z : Z \rightarrow \star$) because $\omega_Z \alpha_Z = id_Z$ and α_Z^* is conservative, so that ω_Z^* preserves binary sums for all Z . \square

Lemma 5.12. *A σ -pointed exact Mal'cev category is algebraically extensive if and only if it is a semi-abelian category with exact pointed base-change functors.*

Proof. This follows from Lemmas 5.10 and 5.11 and the fact that a left exact and regular epimorphism preserving functor between semi-abelian categories is right exact if and only if it preserves binary sums, cf. Section 1.18. \square

For the following lemma a *variety* means a category equipped with a forgetful functor to sets which is monadic and preserves filtered colimits. Every variety is bicomplete, cocomplete, and has finite limits commuting with filtered colimits.

Lemma 5.13. *A semi-abelian variety is (locally) algebraically cartesian closed if and only if it is algebraically distributive (extensive).*

Proof. Since the fibres of a semi-abelian category are semi-abelian, the pointed base-change functors preserve binary sums if and only if they preserve finite colimits, cf. Section 1.18. Since any colimit is a filtered colimit of finite colimits, and pointed base-change functors of a variety preserve filtered colimits, they preserve binary sums if and only if they preserve all colimits. It follows then from Freyd's special adjoint functor theorem that a pointed base-change functor of a semi-abelian variety preserves binary sums if and only if it has a right adjoint. \square

A pointed category \mathbb{D} with pullbacks is called *fibrewise* algebraically cartesian closed (resp. distributive) if for all objects Z of \mathbb{D} the fibres $\text{Pt}_Z(\mathbb{D})$ are algebraically cartesian closed (resp. distributive). This is the case if and only if pointed

base-change along every *split epimorphism* has a right adjoint (resp. preserves binary sums). Any algebraically coherent pointed Mal'cev category is fibrewise algebraically distributive, cf. the proof of Lemma 5.11.

Proposition 5.14. *For a pointed regular Mal'cev category, fibrewise algebraic cartesian closure is preserved under epireflections.*

Proof. Let $(r, s) : X \rightrightarrows Y$ be a point in an epireflective subcategory \mathbb{C} of a fibrewise algebraically cartesian closed regular Mal'cev category \mathbb{D} . Let $f : Y \rightarrow Z$ be a split epimorphism in \mathbb{C} . We shall denote $f_* : \text{Pt}_Y(\mathbb{D}) \rightarrow \text{Pt}_Z(\mathbb{D})$ the *right* adjoint of pointed base-change $f^* : \text{Pt}_Z(\mathbb{D}) \rightarrow \text{Pt}_Y(\mathbb{D})$. Consider the following diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\varepsilon} & \bar{X} & \xrightleftharpoons{\bar{f}} & \bar{X} \\
 \downarrow r & \swarrow s & \downarrow \bar{r} & \searrow \eta_{\bar{X}} & \downarrow \bar{s} \\
 & X & \xleftarrow{I(\varepsilon)} & I(\bar{X}) & \xrightarrow{I(\bar{f})} & I(\bar{X}) \\
 & \swarrow r & \downarrow I(\bar{r}) & \searrow f & \downarrow I(\bar{r}) \\
 Y & \xleftarrow{\quad} & Y & \xrightleftharpoons{\quad} & Z
 \end{array}$$

$\bar{X} \xrightarrow{\quad \bar{\phi} \quad} I(\bar{X})$

where $(\bar{r}, \bar{s}) = f_*(r, s)$, the downward-oriented right square is a pullback, and $\varepsilon : (\bar{r}, \bar{s}) = f^* f_*(r, s) \rightarrow (r, s)$ is the counit at (r, s) . Since \mathbb{D} is a regular Mal'cev category, the epireflection I preserves this pullback of split epimorphisms (cf. Proposition 2.10) so that $I(\bar{X})$ is isomorphic to $f^*(I(\bar{X}))$. Since by adjunction, maps $I(\bar{X}) \rightarrow f_*(X) = \bar{X}$ correspond bijectively to maps $I(\bar{X}) = f^*(I(\bar{X})) \rightarrow X$ there is a unique dotted map $\bar{\phi} : I(\bar{X}) \rightarrow \bar{X}$ such that $\varepsilon \circ f^*(\bar{\phi}) = I(\varepsilon)$.

Accordingly we get $\bar{\phi} \eta_{\bar{X}} = 1_{\bar{X}}$ so that $\eta_{\bar{X}}$ is invertible and hence \bar{X} belongs to \mathbb{C} . This shows that the right adjoint $f_* : \text{Pt}_Y(\mathbb{D}) \rightarrow \text{Pt}_Z(\mathbb{D})$ restricts to a right adjoint $f_* : \text{Pt}_Y(\mathbb{C}) \rightarrow \text{Pt}_Z(\mathbb{C})$ so that \mathbb{C} is fibrewise algebraically cartesian closed. \square

For regular Mal'cev categories, algebraic cartesian closure amounts to the existence of centralizers for all (split) subobjects, see Section 1.21. Part of Proposition 5.14 could thus be reformulated by saying that in this context the existence of centralizers is preserved under epireflections, which can also be proved directly. In a varietal context, Proposition 5.21 also follows from Lemmas 5.13 and 5.20.

Lemma 5.15. *If an algebraically extensive semi-abelian (or homological [5]) category \mathbb{D} has an identity functor of degree $\leq n$, then all its fibres $\text{Pt}_Z(\mathbb{D})$ as well.*

Proof. The kernel functors $(\alpha_Z)^* : \text{Pt}_Z(\mathbb{D}) \rightarrow \mathbb{D}$ are conservative and preserve binary sums. Therefore, the kernel functors preserve the limits $P_{X_1, \dots, X_{n+1}}^{\text{Pt}_Z(\mathbb{D})}$ and the comparison maps $\theta_{X_1, \dots, X_{n+1}}^{\text{Pt}_Z(\mathbb{D})}$. Accordingly, if the identity functor of \mathbb{D} is of degree $\leq n$, then $\alpha_Z^*(\theta_{X_1, \dots, X_{n+1}})$ is invertible, hence so is $\theta_{X_1, \dots, X_{n+1}}^{\text{Pt}_Z(\mathbb{D})}$, for all objects X_1, \dots, X_{n+1} of the fibre $\text{Pt}_Z(\mathbb{D})$. \square

It should be noted that in general neither algebraic extensivity nor local algebraic cartesian closure is preserved under Birkhoff reflection. This is in neat contrast to (fibrewise) algebraic distributivity and algebraic cartesian closure which are both preserved even under epireflection, cf. Proposition 5.14 and Lemma 5.20.

5.16. Duality for pseudo-additive regular Mal'cev categories.

Lemma 5.17. *For any pointed category \mathbb{D} with binary sums and binary products consider the following commutative diagram*

$$\begin{array}{ccc} (X \times Y) + Z & \xrightarrow{\rho_{X,Y,Z}} & X \times (Y + Z) \\ j_X^Y + Z \downarrow p_X^Y + Z & & X \times \iota_Z^Y \downarrow X \times \pi_Z^Y \\ X + Z & \xrightarrow{\theta_{X,Z}} & X \times Z \end{array}$$

in which $\rho_{X,Y,Z}$ is induced by the pair $X \times \iota_Y^Z : X \times Y \rightarrow X \times (Y + Z)$ and $\alpha_X \times \iota_Z^Y : Z \rightarrow X \times (Y + Z)$.

- (1) *pointed base-change $(\omega_X)^* : \mathbb{D} \rightarrow \text{Pt}_X(\mathbb{D})$ preserves binary sums if and only if the upward-oriented square is a pushout for all objects Y, Z ;*
- (2) *pointed cobase-change $(\alpha_Z)_! : \text{Pt}_Z(\mathbb{D}) \rightarrow \mathbb{D}$ preserves binary products if and only if the downward-oriented square is a pullback for all objects X, Y .*

Proof. The left upward-oriented square of the following diagram

$$\begin{array}{ccccc} & & X \times \iota_Y^Z & & \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\ X \times Y & \xrightarrow{\quad} & (X \times Y) + Z & \xrightarrow{\rho_{X,Y,Z}} & X \times (Y + Z) \\ j_X^Y \downarrow p_X^Y & \swarrow \iota_{X \times Y}^Z & j_X^Y + Z \downarrow p_X^Y + Z & & X \times \iota_Z^Y \downarrow X \times \pi_Z^Y \\ X & \xrightarrow{\iota_X^Z} & X + Z & \xrightarrow{\theta_{X,Z}} & X \times Z \\ & \xrightarrow{j_X^Z} & & & \end{array}$$

is a pushout so that the whole upward-oriented rectangle is a pushout (i.e. $(\omega_X)^*$ preserves binary sums) if and only if the right upward-oriented square is a pushout.

The right downward-oriented square of the following diagram

$$\begin{array}{ccccc} & & p_Y^X + Z & & \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\ (X \times Y) + Z & \xrightarrow{\rho_{X,Y,Z}} & X \times (Y + Z) & \xrightarrow{p_{Y+Z}^X} & Y + Z \\ j_X^Y + Z \downarrow p_X^Y + Z & \swarrow X \times \iota_Z^Y & X \times \pi_Z^Y \downarrow \iota_Z^Y & & \pi_Z^Y \\ X + Z & \xrightarrow{\theta_{X,Z}} & X \times Z & \xrightarrow{p_Z^X} & Z \\ & \xrightarrow{\pi_Z^X} & & & \end{array}$$

is a pullback so that the whole downward-oriented rectangle is a pullback (i.e. $(\alpha_Z)_!$ preserves binary products) if and only if the left downward-oriented square is a pullback. \square

Proposition 5.18. *In a σ -pointed regular Mal'cev category \mathbb{D} , pointed base-change $(\omega_Z)^* : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D})$ preserves binary sums for all objects Z as soon as pointed cobase-change $(\alpha_Z)_! : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D})$ preserves binary products for all objects Z . The converse implication holds if \mathbb{D} is pseudo-additive (cf. Definition 4.9).*

Proof. According to the previous lemma pointed cobase-change $(\alpha_X)_!$ preserves binary products if and only if the downward-oriented square is a pullback which implies that the upward-oriented square is a pushout, and hence pointed base-change $(\omega_Z)^*$ preserves binary sums. If \mathbb{D} is pseudo-additive, the comparison map

$\theta_{X,Z}$ is an affine extension. Therefore, the downward-oriented square is a pullback if and only if the upward-oriented square is a pushout, whence the converse. \square

Corollary 5.19. *A σ -pointed exact Mal'cev category is quadratic if and only if it is 2-nilpotent and algebraically distributive.*

Lemma 5.20. *In a σ -pointed regular Mal'cev category, (fibrewise) algebraic distributivity is preserved under epireflections.*

Proof. This follows from Proposition 2.10 which shows that epireflections preserve besides pushouts and binary sums also binary products (in the fibres). \square

Theorem 5.21. *For any algebraically distributive, σ -pointed exact Mal'cev category, the Birkhoff subcategory of 2-nilpotent objects is quadratic.*

Proof. The Birkhoff subcategory is pointed, exact, 2-nilpotent, and algebraically distributive by Lemma 5.20, and hence quadratic by Corollary 5.19. \square

Corollary 5.22 (cf. Cigoli-Gray-Van der Linden [21], Corollary 7.2). *For each object X of an algebraically distributive semi-abelian category, the iterated Huq commutator $[X, [X, X]]$ coincides with the ternary Higgins commutator $[X, X, X]$.*

Proof. Recall (cf. [38, 39]) that $[X, X, X]$ is the direct image of $K[\theta_{X,X,X}]$ under the ternary folding map $X + X + X \rightarrow X$. In general, the iterated Huq commutator $[X, [X, X]]$ is contained in $[X, X, X]$, cf. Corollary 6.13. In a semi-abelian category, the unit of second Birkhoff reflection I^2 takes the form $\eta_X^2 : X \rightarrow X/[X, [X, X]]$, cf. Remark 2.15. Since in the algebraically distributive case, the subcategory of 2-nilpotent objects is quadratic by Theorem 5.21, the image of $[X, X, X]$ in $X/[X, [X, X]]$ is trivial by Corollaries 5.7b and 6.22, whence $[X, [X, X]] = [X, X, X]$. \square

Remark 5.23. The category of groups (resp. Lie algebras) has centralizers for subobjects and is thus algebraically distributive. Therefore, the category of 2-nilpotent groups (resp. Lie algebras) is a quadratic semi-abelian variety.

The reader should observe that although on the level of 2-nilpotent objects there is a perfect symmetry between the property that pointed base-change along terminal maps preserves binary sums and the property that pointed cobase-change along initial maps preserves binary products (cf. Proposition 5.18), only the algebraic distributivity carries over to the category of all groups (resp. Lie algebras) while the algebraic “codistributivity” fails in either of these categories. “Codistributivity” is a quite restrictive property, which is rarely satisfied without assuming 2-nilpotency.

6. IDENTITY FUNCTORS WITH BOUNDED DEGREE

In the previous section we have seen that quadraticity is a slightly stronger property than 2-nilpotency, insofar as it also requires a certain compatibility between binary sum and binary product (cf. Theorem 5.6 and Proposition 5.18). In this last section, we relate n -nilpotency to identity functors of degree $\leq n$.

Any $(n + 1)$ -cube $\Xi_{X_1, \dots, X_{n+1}}$ (cf. Definition 5.1) defines a split natural transformation of n -cubes inducing a natural transformation of split epimorphisms

$$\begin{array}{ccc}
X_1 + \cdots + X_{n+1} & \xrightarrow{\theta_{X_1, \dots, X_n}^{+X_{n+1}}} & P_{X_1, \dots, X_n}^{+X_{n+1}} \\
\pi_{\hat{X}_{n+1}} \updownarrow & \textcircled{n} & P_{X_1, \dots, X_n}^{+\omega X_{n+1}} \updownarrow P_{X_1, \dots, X_n}^{+\alpha X_{n+1}} \\
X_1 + \cdots + X_n & \xrightarrow{\theta_{X_1, \dots, X_n}} & P_{X_1, \dots, X_n}
\end{array}$$

the comparison map of which may be identified with the comparison map

$$\theta_{X_1, \dots, X_{n+1}} : X_1 + \cdots + X_{n+1} \rightarrow P_{X_1, \dots, X_{n+1}}$$

of the given $(n+1)$ -cube. In particular, our category has an identity functor of degree $\leq n$ if and only if square (n) is a pullback square for all objects X_1, \dots, X_{n+1} .

Proposition 6.1. *In a σ -pointed regular Mal'cev category, the comparison map $\theta_{X_1, \dots, X_{n+1}}$ is a regular epimorphism with kernel relation $R[\pi_{\hat{X}_1}] \cap \cdots \cap R[\pi_{\hat{X}_{n+1}}]$.*

Proof. This follows by induction on n like in the proof of Proposition 5.2. \square

Remark 6.2. The previous proposition shows that in a σ -pointed regular Mal'cev category, the intersection of the kernel relations of the contraction maps may be considered as the “total kernel relation” of the cube. This parallels the more elementary fact that the total-kernel definition of the cross-effects $cr_{n+1}(X, \dots, X_{n+1})$ coincides with the kernel-intersection definition of Hartl-Loiseau [38] and Hartl-Van der Linden [39]. In particular, in any σ -pointed regular Mal'cev category, the image of the morphism $r_{id} : X_1 + \cdots + X_{n+1} \rightarrow \prod_{i=1}^{n+1} X_1 + \cdots + \widehat{X_i} + \cdots + X_{n+1}$ coincides with the limit $P_{X_1, \dots, X_{n+1}}$ of the punctured $(n+1)$ -cube.

We already mentioned that these kernel intersections are strictly dual to the $(n+1)$ -fold smash products of Carboni-Janelidze [17]. An alternative way to describe the duality between cross-effects and smash-products is to consider the limit construction P_{X_1, \dots, X_n} as the dual of the so-called *fat wedge* T^{X_1, \dots, X_n} , cf. [42]. Set-theoretically, the fat wedge is the subobject of the product $X_1 \times \cdots \times X_n$ formed by the n -tuples having at least one coordinate at a base-point. If base-point inclusions behave “well” with respect to cartesian product, the fat wedge is given by a colimit construction, strictly dual to the limit construction defining P_{X_1, \dots, X_n} . The n -fold smash-product $X_1 \wedge \cdots \wedge X_n$ is then the cokernel of the monomorphism $T^{X_1, \dots, X_n} \hookrightarrow X_1 \times \cdots \times X_n$ while the n -th cross-effect $cr_n(X_1, \dots, X_n)$ is the kernel of the regular epimorphism $X_1 + \cdots + X_n \twoheadrightarrow P_{X_1, \dots, X_n}$.

The cubical cross-effects are just the algebraic version of Goodwillie's homotopical cross-effects [29, pg. 676]. Nevertheless, for functors taking values in abelian categories, the cubical cross-effects agree with the original cross-effects of Eilenberg-Mac Lane [24, pg. 77]. Indeed, by [24, Theorems 9.1 and 9.6], for a based functor $F : \mathbb{D} \rightarrow \mathbb{E}$ from a σ -pointed category $(\mathbb{D}, +, \star_{\mathbb{D}})$ to an abelian category $(\mathbb{E}, \oplus, 0_{\mathbb{E}})$, the latter are completely determined by the following *decomposition formula*

$$F(X_1 + \cdots + X_n) \cong \bigoplus_{1 \leq k \leq n} \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} cr_k^F(X_{i_1}, \dots, X_{i_k})$$

for any objects X_1, \dots, X_n in \mathbb{D} . It suffices thus to show that the cubical cross-effects satisfy the decomposition formula if values are taken in an abelian category.

For $n = 2$ we get $P_{X_1, X_2}^F = F(X_1) \oplus F(X_2)$ from which it follows that $\theta_{X_1, X_2}^F : F(X_1 + X_2) \twoheadrightarrow F(X_1) \oplus F(X_2)$ is a *split* epimorphism. Henceforth, we get the asserted isomorphism $F(X_1 + X_2) \cong F(X_1) \oplus F(X_2) \oplus cr_2^F(X_1, X_2)$.

The 3-cube $F(\Xi_{X_1, X_2, X_3})$ induces a natural transformation of split epimorphisms

$$\begin{array}{ccc} F(X_1 + X_2 + X_3) & \xrightarrow{\quad} & P_3 \\ \updownarrow & & \updownarrow \\ F(X_1 + X_2) & \xrightarrow[\theta_{X_1, X_2}^F]{} & F(X_1) \oplus F(X_2) \end{array}$$

in which P_3 is isomorphic to $F(X_1) \oplus F(X_2) \oplus F(X_3) \oplus cr_2^F(X_1, X_3) \oplus cr_2^F(X_2, X_3)$. From this, we get for $P_{X_1, X_2, X_3}^F = F(X_1 + X_2) \times_{F(X_1) \oplus F(X_2)} P_3$ the formula

$$P_{X_1, X_2, X_3}^F \cong F(X_1) \oplus F(X_2) \oplus F(X_3) \oplus cr_2^F(X_1, X_2) \oplus cr_2^F(X_1, X_3) \oplus cr_2^F(X_2, X_3)$$

so that θ_{X_1, X_2, X_3}^F is again a *split* epimorphism inducing the asserted decomposition of $F(X_1 + X_2 + X_3)$. The same scheme keeps on for all positive integers n . \square

Definition 6.3. An object X of a σ -pointed regular Mal'cev category will be called *n-folded* if the folding map δ_{n+1}^X factors through the comparison map $\theta_{X, \dots, X}$

$$\begin{array}{ccc} X + \dots + X & \xrightarrow{\theta_{X, \dots, X}} & P_{X, \dots, X} \\ & \searrow \delta_{n+1}^X & \swarrow m_X \\ & X & \end{array}$$

i.e. if the folding map δ_{n+1}^X annihilates the kernel relation $R[\theta_{X, \dots, X}]$.

An object X is *1-folded* if and only if the identity of X commutes with itself. In a σ -pointed regular Mal'cev category this is the case if and only if X is an abelian group object, cf. the proof of Proposition 4.2.

Remark 6.4. In a semi-abelian (or homological [5]) category, an object X is *n-folded* if and only if the image of the kernel $K[\theta_{X, \dots, X}]$ under $\delta_{n+1}^X : X + \dots + X \rightarrow X$ is trivial. In a varietal context, this can be expressed in more combinatorial terms.

For instance, a *group* X is *n-folded* if and only if $(n+1)$ -*reducible elements* of X are *trivial*. An element $w \in X$ is called $(n+1)$ -reducible if there is an element v in the *free group* $F(X \sqcup \dots \sqcup X)$ on $n+1$ copies of X (viewed as a set) such that

(a) w is the image of v under the composite map

$$F(\overbrace{X \sqcup \dots \sqcup X}^{n+1}) \cong F(X) + \dots + F(X) \twoheadrightarrow \overbrace{X + \dots + X}^{n+1} \xrightarrow{\delta_{n+1}^X} X$$

(b) for each of the $n+1$ contractions $\pi_i^{F(X)} : F(X)^{+(n+1)} \twoheadrightarrow F(X)^{+n}$, cf. Section 6.10, the image $\pi_i^{F(X)}(v)$ maps to the neutral element of X under

$$\overbrace{F(X) + \dots + F(X)}^n \twoheadrightarrow \overbrace{X + \dots + X}^n \xrightarrow{\delta_n^X} X.$$

Indeed, since the evaluation map $F(X) \rightarrow X$ is a regular epimorphism, and the evaluation maps in (a) and (b) are compatible with the contraction maps $\pi_i : X^{+(n+1)} \twoheadrightarrow X^{+n}$, Proposition 6.1, Section 6.10 and Corollary 6.22 imply that we get in this way precisely the image of the kernel $K[\theta_{X, \dots, X}]$ under folding δ_{n+1}^X .

Any product of commutators $\prod_{i=1}^k [x_i, y_i] = \prod_{i=1}^k x_i y_i x_i^{-1} y_i^{-1}$ in X is 2-reducible by letting the x_i (resp. y_i) belong to the first (resp. second) copy of X . Conversely, a direct computation shows that any 2-reducible element of X can be rewritten

as a product of commutators of X . This recovers in a combinatorial way the aforementioned fact that X is abelian (i.e. 1-nilpotent) if and only if X is 1-folded.

The relationship between n -nilpotency and n -foldedness is more subtle, closely related to the cross-effects of the identity functor (cf. Theorem 6.8). For groups and Lie algebras the two concepts coincide (cf. Theorem 6.26c) but, though in general any n -folded object is n -nilpotent, the converse might be wrong (cf. Section 6.28).

Proposition 6.5. *Let $F : \mathbb{D} \rightarrow \mathbb{E}$ be a based functor between σ -pointed categories and assume that \mathbb{E} is a regular Mal'cev category.*

- (a) *If F is of degree $\leq n$ then F takes values in n -folded objects of \mathbb{E} ;*
- (b) *If F preserves binary sums and takes values in n -folded objects of \mathbb{E} then F is of degree $\leq n$;*
- (c) *The identity functor of \mathbb{E} is of degree $\leq n$ if and only if all objects of \mathbb{E} are n -folded.*

Proof. Clearly, (c) follows from (a) and (b). For (a) note that $\delta_{n+1}^{F(X)}$ factors through $F(\delta_{n+1}^X)$, and that by definition of a functor of degree $\leq n$, the comparison map $\theta_{X, \dots, X}^F$ is invertible so that $F(\delta_{n+1}^X)$ gets identified with $m_{F(X)}$.

For (b) observe first that preservation of binary sums yields the isomorphisms $P_{X_1, \dots, X_{n+1}}^F \cong P_{F(X_1), \dots, F(X_{n+1})}$ and $\theta_{X_1, \dots, X_{n+1}}^F \cong \theta_{F(X_1), \dots, F(X_{n+1})}$. We shall show that if moreover F takes values in n -folded objects of \mathbb{E} then $\theta_{X_1, \dots, X_{n+1}}^F$ is invertible for all $(n+1)$ -tuples (X_1, \dots, X_{n+1}) of objects of \mathbb{D} .

Consider any family $(f_i : X_i \rightarrow T)_{1 \leq i \leq n+1}$ of morphisms in \mathbb{E} , and let $\phi = \delta_{n+1}^T \circ (f_1 + \dots + f_{n+1}) : X_1 + \dots + X_{n+1} \rightarrow T$ be the induced map. We have the following factorization of $F(\phi)$ through $\theta_{X_1, \dots, X_{n+1}}^F$:

$$\begin{array}{ccc}
 F(X_1) + \dots + F(X_{n+1}) & \xrightarrow{\theta_{F(X_1), \dots, F(X_{n+1})}^F} & P_{F(X_1), \dots, F(X_{n+1})} \\
 \downarrow F(f_1) + \dots + F(f_{n+1}) & & \downarrow P_{F(f_1), \dots, F(f_{n+1})} \\
 F(T) + \dots + F(T) & \xrightarrow{\theta_{F(T), \dots, F(T)}^F} & P_{F(T), \dots, F(T)} \\
 \searrow \delta_{n+1}^T & & \swarrow m_{F(T)} \\
 & F(T) &
 \end{array}$$

In particular, if $T = X_1 + \dots + X_{n+1}$ and f_i is the inclusion of the i th summand, we get a retraction of $\theta_{F(X_1), \dots, F(X_{n+1})}^F$ which accordingly is a monomorphism. Since $\theta_{F(X_1), \dots, F(X_{n+1})}^F$ is also a regular epimorphism, it is invertible. \square

Proposition 6.6. *The full subcategory $\text{Fld}^n(\mathbb{E})$ of n -folded objects of a σ -pointed regular Mal'cev category \mathbb{E} is closed under products, subobjects and quotients.*

Proof. For any two n -folded objects X and Y the following diagram

$$\begin{array}{ccc}
 (X \times Y) + \cdots + (X \times Y) & \xrightarrow{\theta_{X \times Y, \dots, X \times Y}} & P_{X \times Y, \dots, X \times Y} \\
 \downarrow & & \downarrow \\
 (X + \cdots + X) \times (Y + \cdots + Y) & \xrightarrow{\theta_{X, \dots, X} \times \theta_{Y, \dots, Y}} & P_{X, \dots, X} \times P_{Y, \dots, Y} \\
 \searrow \delta_{n+1}^X \times \delta_{n+1}^Y & & \swarrow m_X \times m_Y \\
 & X \times Y &
 \end{array}$$

induces the required factorization of $\delta_{n+1}^{X \times Y}$ through $\theta_{X \times Y, \dots, X \times Y}$.

For a subobject $n : U \rightarrow X$ of an n -folded object X consider the diagram

$$\begin{array}{ccc}
 U + \cdots + U & \xrightarrow{\theta_{U, \dots, U}} & P_{U, \dots, U} \\
 \downarrow \delta_{n+1}^U & \searrow \theta & \downarrow P_{n, \dots, n} \\
 & W & \\
 & \swarrow \nu & \\
 & P_{U, \dots, U} & \\
 & \downarrow & \\
 & X + \cdots + X & \xrightarrow{\theta_{X, \dots, X}} P_{X, \dots, X} \\
 \downarrow \delta_{n+1}^X & \swarrow m_X & \\
 U & \xrightarrow{n} & X
 \end{array}$$

in which the dotted quadrangle is a pullback. The commutativity of the diagram induces a morphism θ such that $\nu\theta = \theta_{U, \dots, U}$. Since $\theta_{U, \dots, U}$ is a regular epimorphism, the monomorphism ν is invertible, whence the desired factorization.

Finally, for a regular epimorphism $f : X \rightarrow Y$ with n -folded domain X consider the following diagram

$$\begin{array}{ccccc}
 X + \cdots + X & \xrightarrow{f + \cdots + f} & Y + \cdots + Y & & \\
 \downarrow \delta_{n+1}^X & \searrow \theta_{X, \dots, X} & \downarrow \delta_{n+1}^Y & \searrow \theta_{Y, \dots, Y} & \\
 & P_{X, \dots, X} & \xrightarrow{P_{f, \dots, f}} & P_{Y, \dots, Y} & \\
 \downarrow m_X & & & \downarrow m_Y & \\
 X & \xrightarrow{f} & Y & &
 \end{array}$$

in which the existence of the dotted arrow has to be shown. According to Lemma 6.7 the induced morphism on kernel relations

$$R(f + \cdots + f, P_{f, \dots, f}) : R[\theta_{X, \dots, X}] \rightarrow R[\theta_{Y, \dots, Y}]$$

is a regular epimorphism. A diagram chase shows then that δ_{n+1}^Y annihilates $R[\theta_{Y, \dots, Y}]$ whence the required factorization of δ_{n+1}^Y through $\theta_{Y, \dots, Y}$. \square

Lemma 6.7. *In a σ -pointed regular Mal'cev category, any finite family of regular epimorphisms $f_i : X_i \rightarrow Y_i$ ($i = 1, \dots, n$) induces a regular epimorphism on kernel relations $R(f_1 + \cdots + f_n, P_{f_1, \dots, f_n}) : R[\theta_{X_1, \dots, X_n}] \rightarrow R[\theta_{Y_1, \dots, Y_n}]$.*

Proof. Since regular epimorphisms compose (in any regular category) it suffices to establish the assertion under the assumption $f_i = 1_{X_i}$ for $i = 2, \dots, n$. Moreover we can argue by induction on n since for $n = 1$ the comparison map is the terminal

map $\theta_X : X \rightarrow \star$ and a binary product of regular epimorphisms is a regular epimorphism. Assume now that the statement is proved for $n - 1$ morphisms. Using the isomorphism of kernel relations

$$R[\theta_{X_1, \dots, X_{n-1}, X_n}] \cong R[R[\theta_{X_1, \dots, X_{n-1}}^{+X_n}] \rightarrow R[\theta_{X_1, \dots, X_{n-1}}]]$$

and Proposition 1.9 it suffices then to show that the following by $f_1 : X_1 \rightarrow Y_1$ induced commutative square

$$\begin{array}{ccc} R[\theta_{X_1, X_2, \dots, X_{n-1}}^{+X_n}] & \longrightarrow & R[\theta_{Y_1, X_2, \dots, X_{n-1}}^{+X_n}] \\ \updownarrow & & \updownarrow \\ R[\theta_{X_1, X_2, \dots, X_{n-1}}] & \longrightarrow & R[\theta_{Y_1, X_2, \dots, X_{n-1}}] \end{array}$$

is a downward-oriented *regular* pushout. This in turn follows from Corollary 1.10 since the vertical arrows above are split epimorphisms by construction and the horizontal arrows are regular epimorphisms by induction hypothesis. \square

Theorem 6.8. *The full subcategory $\text{Fld}^n(\mathbb{E})$ of n -folded objects of a σ -pointed exact Mal'cev category \mathbb{E} is a reflective Birkhoff subcategory. The associated Birkhoff reflection $J^n : \mathbb{E} \rightarrow \text{Fld}^n(\mathbb{E})$ is the universal endofunctor of \mathbb{E} of degree $\leq n$.*

Proof. Observe that the second assertion is a consequence of the first and of Proposition 6.5a-b. In virtue of Proposition 6.6 it suffices thus to construct the reflection $J^n : \mathbb{E} \rightarrow \text{Fld}^n(\mathbb{E})$. The latter is obtained by the following pushout

$$\begin{array}{ccc} X + \dots + X & \xrightarrow{\theta_{X, \dots, X}} & P_{X, \dots, X} \\ \delta_{n+1}^X \downarrow & & \downarrow \mu_n X \\ X & \xrightarrow{\epsilon_n X} & J^n(X) \end{array}$$

which is regular by Corollary 1.11 so that $J^n(X) = X/H_{n+1}[X]$ where $H_{n+1}[X]$ is the direct image of $R[\theta_{X, \dots, X}]$ under the folding map δ_{n+1}^X . We will show that $J^n(X)$ is n -folded and that any morphism $X \rightarrow T$ with n -folded codomain T factors uniquely through $J^n(X)$. For this, consider the following diagram

$$\begin{array}{ccccccc} H_{n+1}[X] + \dots + H_{n+1}[X] & \xrightarrow{\theta_{H_{n+1}[X], \dots, H_{n+1}[X]}} & P_{H_{n+1}[X], \dots, H_{n+1}[X]} & & & & \\ \downarrow \delta_{n+1}^{H_{n+1}[X]} & \searrow p_1 + \dots + p_1 & \downarrow P_{p_0, \dots, p_0} & \searrow P_{p_1, \dots, p_1} & & & \\ & p_0 + \dots + p_0 & X + \dots + X & \xrightarrow{\theta_{X, \dots, X}} & P_{X, \dots, X} & \xrightarrow{P_{\epsilon_n X, \dots, \epsilon_n X}} & P_{J^n(X), \dots, J^n(X)} \\ & & \downarrow \delta_{n+1}^X & & \downarrow \mu_n X & & \\ H_{n+1}[X] & \xrightarrow[p_0]{p_1} & X & \xrightarrow{\epsilon_n X} & J^n(X) & \xleftarrow{\dots} & P_{J^n(X), \dots, J^n(X)} \end{array}$$

in which the existence of the dotted arrow has to be shown. By Lemma 6.9 $P_{J^n(X), \dots, J^n(X)}$ is the coequalizer of the reflexive pair $(P_{p_0, \dots, p_0}, P_{p_1, \dots, p_1})$. It suffices thus to check that $\mu_n X$ coequalizes the same pair. This follows by precomposition with the regular epimorphism $\theta_{H_{n+1}[X], \dots, H_{n+1}[X]}$ using the commutativity of the previous diagram.

For the universal property of $\epsilon_n X : X \rightarrow J^n(X)$ let us consider a morphism $f : X \rightarrow T$ with n -folded codomain T . By construction of $J^n(X)$, the following commutative diagram

$$\begin{array}{ccccc}
 X + \cdots + X & \xrightarrow{\theta_{X,\dots,X}} & P_{X,\dots,X} & \xrightarrow{P_{f,\dots,f}} & \\
 \downarrow \delta_{n+1}^X & \searrow f+\cdots+f & \searrow & \searrow & \\
 & T + \cdots + T & \xrightarrow{\theta_{T,\dots,T}} & P_{T,\dots,T} & \\
 & \searrow \delta_{n+1}^T & \searrow & \searrow & \\
 X & \xrightarrow{f} & T & \xleftarrow{m_T} &
 \end{array}$$

induces the desired factorization. \square

Lemma 6.9. *In a σ -pointed exact Mal'cev category, the functor $P_{X_1,\dots,X_{n+1}}$ preserves reflexive coequalizers in each of its $n+1$ variables.*

Proof. By exactness, it suffices to show that P preserves regular epimorphisms in each variable, and that for a regular epimorphism $f_i : X_i \rightarrow X'_i$ the induced map on kernel relations $P_{X_1,\dots,R[f_i],\dots,X_{n+1}} \rightarrow R[P_{X_1,\dots,f_i,\dots,X_{n+1}}]$ is a regular epimorphism as well. By symmetry, it is even sufficient to do so for the first variable.

We shall argue by induction on n (since for $n = 1$ there is nothing to prove) and consider the following downward-oriented pullback diagram

$$\begin{array}{ccc}
 P_{X_1,\dots,X_n,X_{n+1}} & \xrightarrow{\quad} & P_{X_1,\dots,X_n}^{+X_{n+1}} \\
 \Downarrow & & \Downarrow \\
 X_1 + \cdots + X_n & \xrightarrow{\theta_{X_1,\dots,X_n}} & P_{X_1,\dots,X_n}
 \end{array}$$

which derives from square (n) of the beginning of this section. By induction hypothesis, the two lower corners and the upper right corner are functors preserving regular epimorphisms in the first variable. It follows then from the cogluing lemma (cf. the proof of Theorem 6.26a) and Corollary 1.10 that the upper left corner also preserves regular epimorphisms in the first variable.

It remains to be shown that for $f : X_1 \rightarrow X'_1$ we get an induced regular epimorphism on kernel relations. For this we denote by F, G, H the functors induced on the lower left, lower right and upper right corners, and consider the following commutative diagram

$$\begin{array}{ccccc}
 P(R[f]) & \xrightarrow{\quad} & H(R[f]) & \xrightarrow{\rho_H} & \\
 \swarrow \rho_P & \searrow \rho_P & \swarrow t_{R[f]} & \searrow \rho_H & \\
 & R[P(f)] & \xrightarrow{\quad} & R[H(f)] & \\
 \swarrow \rho_F & \searrow \rho_F & \swarrow \theta_{R[f]} & \searrow \rho_G & \\
 F(R[f]) & \xrightarrow{\quad} & G(R[f]) & \xrightarrow{R(t_X)} & \\
 \swarrow \rho_F & \searrow \rho_F & \swarrow \rho_G & \searrow \rho_G & \\
 & R[F(f)] & \xrightarrow{R(\theta_X)} & R[G(f)] &
 \end{array}$$

in which the back vertical square is a downward-oriented pullback by definition of P . By commutation of limits the front vertical square is a downward oriented pullback as well. Again, according to the cogluing lemma and Corollary 1.10, the

induced arrow ρ_P is then a regular epimorphisms, since ρ_F , ρ_G and ρ_H are so by induction hypothesis. \square

6.10. Higgins commutator relations and their normalization. –

We shall now concentrate on the case $X = X_1 = X_2 = \cdots = X_{n+1}$. Accordingly, we abbreviate the $n + 1$ “contractions” as follows:

$$\pi_{\hat{X}_i} = \pi_i : \overbrace{X + \cdots + X}^{n+1} \rightarrow \overbrace{X + \cdots + X}^n, \quad i = 1, \dots, n + 1.$$

Proposition 6.1 reads then $R[\theta_{X,\dots,X}] = R[\pi_1] \cap \cdots \cap R[\pi_{n+1}]$.

We denote the direct image of the kernel relation $R[\theta_{X,\dots,X}]$ under the folding map $\delta_{n+1}^X : X + \cdots + X \rightarrow X$ by a single bracket $[\nabla_X, \dots, \nabla_X]$ of length $n + 1$ and call it the $(n + 1)$ -ary Higgins commutator relation on X .

The proof of Theorem 6.8 shows that in σ -pointed exact Mal'cev categories the universal n -folded quotient $J^n(X)$ of an object X is obtained by quotienting out the $(n + 1)$ -ary Higgins commutator relation. The binary Higgins commutator relation coincides with the Smith commutator $[\nabla_X, \nabla_X]$ (cf. Section 1.1, Corollary 1.11 and Proposition 4.2) which ensures consistency of our notation.

Recall that in a pointed category the *normalization* of an effective equivalence relation R on X is the kernel of its quotient map $X \twoheadrightarrow X/R$. In pointed exact Mal'cev categories normalization commutes with direct image, cf. Corollary 1.11.

The direct image of the kernel $K[\theta_{X,\dots,X}]$ under the $(n + 1)$ st folding map δ_{n+1}^X is commonly denoted by a single bracket $[X, \dots, X]$ of length $n + 1$ and called the *Higgins commutator of X of length $n + 1$* , see [38, 39, 41, 54] for further details. In σ -pointed exact Mal'cev categories, these Higgins commutators are thus the normalizations of the Higgins commutator relations of same length.

Proposition 6.11. *In a σ -pointed exact Mal'cev category, the image of $R[\theta_{X,X,X}]$ under $\delta_2^X + 1_X$ is the kernel relation of the pushout of $\theta_{X,X,X}$ along $\delta_2^X + 1_X$*

$$\begin{array}{ccc} X + X + X & \xrightarrow{\theta_{X,X,X}} & P_{X,X,X} \\ \delta_2^X + 1_X \downarrow & & \downarrow \\ X + X & \xrightarrow{\zeta_{X,X}^X} & J_{X,X}^X \end{array}$$

which may be computed as an intersection: $R[\zeta_{X,X}^X] = [R[\pi_1], R[\pi_1]] \cap R[\pi_2]$.

In particular, we get the inclusion $[\nabla_X, [\nabla_X, \nabla_X]] \subset [\nabla_X, \nabla_X, \nabla_X]$.

Proof. By Corollary 1.11, the pushout is regular so that the first assertion follows from Proposition 1.9. Consider the following diagram

$$\begin{array}{ccccc} & X + X + X & \xrightarrow{\delta_2^X + 1_X} & X + X & \xrightarrow{\eta^1(\pi_1)} \\ & \searrow \theta_{X,X,X}^+ & & \nearrow \eta^1(X) & \\ & (X + X) \times_X (X + X) & \xrightarrow{\pi_2} & I^1(\pi_1) & \\ \pi_3 \nearrow & & & & \\ X + X & \xrightarrow{\delta_2^X} & X & \xrightarrow{\eta^1(X)} & I^1(X) \\ \theta_{X,X} \searrow & & & \nearrow f' & \\ & X \times X & \xrightarrow{\quad} & & \end{array}$$

in which top and bottom are regular pushouts by Corollary 1.11. The bottom square constructs the associated abelian object $I^1(X)$ of X , while the top square constructs the associated abelian object $I^1(\pi_1)$ of $\pi_1 : X + X \rightrightarrows X$ in the fibre over X . The upward oriented back and front faces are pushouts of split monomorphisms. The left face is a specialization of square (2) just before Proposition 5.2. We can therefore apply Corollary 1.25 and we get diagram

$$\begin{array}{ccccc}
 X + X + X & \xrightarrow{\delta_2^X + 1_X} & X + X & & \\
 \theta_{X,X,X} \searrow & & \zeta_{X,X}^X \searrow & & \\
 & P_{X,X,X} & \xrightarrow{\quad} & J_{X,X}^X & \\
 \pi_3 \swarrow & & \pi_2 \swarrow & & \\
 X + X & \xrightarrow{\delta_2^X} & X & & \\
 & \nearrow & \nearrow & &
 \end{array}$$

in which the kernel relation of the regular epimorphism $\zeta_{X,X}^X$ is given by

$$R[\eta^1(\pi_1)] \cap R[\pi_2] = [R[\pi_1], R[\pi_1]] \cap R[\pi_2].$$

For the second assertion, observe first that the ternary folding map δ_3 may be identified with the composition $\delta_2^X \circ (\delta_2^X + 1_X)$. Therefore, the ternary Higgins commutator relation $[\nabla_X, \nabla_X, \nabla_X]$ is the direct image under $\delta_2^X : X + X \rightarrow X$ of the kernel relation of $\zeta_{X,X}^X$. Now we have the following chain of inclusions, where for shortness we write $R_1 = R[\pi_1]$, $R_2 = R[\pi_2]$, $R_{12} = R_1 \cap R_2$:

$$[R_{12}, [R_{12}, R_{12}]] \subset R_{12} \cap [R_{12}, R_{12}] \subset R_2 \cap [R_1, R_1].$$

By exactness, the direct image of the leftmost relation is $[\nabla_X, [\nabla_X, \nabla_X]]$, while the direct image of the right most relation is $[\nabla_X, \nabla_X, \nabla_X]$. \square

Proposition 6.12. *In a σ -pointed exact Mal'cev category, the image of $R[\theta_{X,\dots,X}]$ under $\delta_{n-1}^X + 1_X$ is the kernel relation of the pushout of $\theta_{X,\dots,X}$ along $\delta_{n-1}^X + 1_X$*

$$\begin{array}{ccc}
 X + \dots + X & \xrightarrow{\theta_{X,\dots,X}} & P_{X,\dots,X} \\
 \delta_{n-1}^X + 1_X \downarrow & & \downarrow \\
 X + X & \xrightarrow{\zeta_{X,\dots,X}^X} & J_{X,\dots,X}^X
 \end{array}$$

which may be computed as an intersection $R[\zeta_{X,\dots,X}^X] = \overbrace{[R[\pi_1], \dots, R[\pi_1]]}^{n-1} \cap R[\pi_2]$.

In particular, we get the inclusion $[\nabla_X, \overbrace{[\nabla_X, \dots, \nabla_X]}^{n-1}] \subset \overbrace{[\nabla_X, \dots, \nabla_X]}^n$.

Proof. The first assertion follows from Proposition 1.9 and the following diagram

$$\begin{array}{ccccc}
 X + \cdots + X & \xrightarrow{\delta_{n-1}^X + 1_X} & X + X & \xrightarrow{q} & \frac{X+X}{[\nabla_{\pi_1}, \dots, \nabla_{\pi_1}]} \\
 \uparrow \pi_n & \searrow \theta_{X, \dots, X}^+ & \uparrow & & \uparrow \\
 & P_{X, \dots, X}^+ & \xrightarrow{\pi_2} & & \\
 X + \cdots + X & \xrightarrow{\delta_{n-1}^X} & X & \xrightarrow{\pi_2} & \frac{X+X}{[\nabla_{\pi_1}, \dots, \nabla_{\pi_1}]} \\
 \searrow \theta_{X, \dots, X} & \swarrow & \downarrow & & \downarrow \\
 & P_{X, \dots, X} & \xrightarrow{\pi_2} & & \\
 & & X/[\nabla_X, \dots, \nabla_X] & &
 \end{array}$$

in which top and bottom are regular pushouts by Corollary 1.11. The bottom square constructs the quotient of X by the $(n-1)$ -ary Higgins commutator relation $[\nabla_X, \dots, \nabla_X]$. The top square constructs the quotient of $\pi_1 : X + X \rightrightarrows X$ by the $(n-1)$ -ary Higgins commutator relation $[\nabla_{\pi_1}, \dots, \nabla_{\pi_1}]$ in the fibre over X . The upward oriented back and front faces are pushouts of split monomorphisms. The left face is a specialization of square (n) of the beginning of this section. We can therefore apply Corollary 1.25 and we get the following diagram

$$\begin{array}{ccccc}
 X + \cdots + X & \xrightarrow{\delta_{n-1}^X + 1_X} & X + X & \xrightarrow{\zeta_{X, \dots, X}^X} & J_{X, \dots, X}^X \\
 \uparrow \pi_n & \searrow \theta_{X, \dots, X} & \uparrow \pi_2 & & \uparrow \\
 & P_{X, \dots, X} & \xrightarrow{\pi_2} & & \\
 X + \cdots + X & \xrightarrow{\delta_{n-1}^X} & X & \xrightarrow{\zeta_{X, \dots, X}^X} & J_{X, \dots, X}^X \\
 \searrow & \swarrow & \downarrow & & \downarrow \\
 & & X & &
 \end{array}$$

in which the kernel relation of the regular epimorphism $\zeta_{X, \dots, X}^X$ is given by

$$R[q] \cap R[\pi_2] = [\nabla_{\pi_1}, \dots, \nabla_{\pi_1}] \cap R[\pi_2] = [R[\pi_1], \dots, R[\pi_1]] \cap R[\pi_2].$$

Since $\delta_n^X = \delta_2^X \circ (\delta_{n-1}^X + 1_X)$, the proof of the second assertion is completely analogous to the proof of the corresponding part of Proposition 6.11. \square

Corollary 6.13. *For any object X of a σ -pointed exact Mal'cev category, the iterated Smith commutator $[\nabla_X, [\nabla_X, [\nabla_X, \dots, [\nabla_X, \nabla_X] \dots]]$ is a subobject of the Higgins commutator relation $[\nabla_X, \dots, \nabla_X]$ of same length.*

In a semi-abelian category, the iterated Huq commutator $[X, [X, \dots, [X, X] \dots]]$ is a subobject of the Higgins commutator $[X, \dots, X]$ of same length.

Proof. The first statement follows inductively from Propositions 6.11 and 6.12.

The second statement follows from the first and the fact that in a semi-abelian category, the iterated Huq commutator is the normalization of the iterated Smith commutator by Remark 2.15 and [33, Proposition 2.2]. \square

Proposition 6.14. *In a σ -pointed exact Mal'cev category \mathbb{E} , each n -folded object is an n -nilpotent object, i.e. $\text{Fld}^n(\mathbb{E}) \subset \text{Nil}^n(\mathbb{E})$. In particular, endofunctors of degree $\leq n$ take values in n -nilpotent objects.*

Proof. The second assertion follows from the first and from Proposition 6.5. For any n -folded object X , the $(n+1)$ -ary Higgins commutator relation of X is discrete

and hence, by Corollary 6.13, the iterated Smith commutator of same length is discrete as well. By an iterated application of Theorem 2.14 and Proposition 1.6, this iterated Smith commutator is the kernel relation of $\eta_X^n : X \twoheadrightarrow I^n(X)$, and hence $X \cong I^n(X)$, i.e. X is n -nilpotent. \square

The following theorem generalizes Theorem 4.5 to all positive integers.

Theorem 6.15. *For a σ -pointed exact Mal'cev category \mathbb{D} such that the identity functor of $\text{Nil}^{n-1}(\mathbb{D})$ is of degree $\leq n-1$, the following properties are equivalent:*

- (a) *all objects are n -nilpotent;*
- (b) *for all objects X_1, \dots, X_n , the map θ_{X_1, \dots, X_n} is an affine extension;*
- (c) *for all objects X_1, \dots, X_n , the map θ_{X_1, \dots, X_n} is a central extension.*

Proof. For an n -nilpotent category \mathbb{D} , the Birkhoff reflection $I^{n-1} : \mathbb{D} \rightarrow \text{Nil}^{n-1}(\mathbb{D})$ is a central reflection. Since all limits involved in the construction of P_{X_1, \dots, X_n} are preserved under I^{n-1} by an iterative application of Proposition 2.10, we get $I^{n-1}(\theta_{X_1, \dots, X_n}) = \theta_{I^{n-1}(X_1), \dots, I^{n-1}(X_n)}$. Since by assumption the identity functor of $\text{Nil}^{n-1}(\mathbb{D})$ is of degree $\leq n-1$, the latter comparison map is invertible so that by Theorem 3.5, the comparison map θ_{X_1, \dots, X_n} is an affine extension, i.e. (a) implies (b). By Proposition 3.4, (b) implies (c).

Specializing (c) to the case $X = X_1 = X_2 = \dots = X_n$ we get the following commutative diagram

$$\begin{array}{ccccc}
 R[\theta_{X, \dots, X}] & \xrightarrow{\quad} & X + \dots + X & \xrightarrow{\theta_{X, \dots, X}} & P_{X, \dots, X} \\
 \downarrow & & \downarrow \delta_n^X & & \downarrow \\
 [\nabla_X, \dots, \nabla_X] & \xrightarrow{\quad} & X & \xrightarrow{\quad} & X/[\nabla_X, \dots, \nabla_X]
 \end{array}$$

in which the right square is a regular pushout by Corollary 1.11 so that the lower row represents the kernel relation of a central extension. We have already seen that the iterated Smith commutator $[\nabla_X, [\nabla_X, [\nabla_X, \dots, [\nabla_X, \nabla_X] \dots]]$ of length n is the kernel relation of the unit $\eta^{n-1}(X) : X \twoheadrightarrow I^{n-1}(X)$ of the $(n-1)$ st Birkhoff reflection. Corollary 6.13 implies thus that this unit is a central extension as well so that \mathbb{D} is n -nilpotent, i.e. (c) implies (a). \square

Definition 6.16. *A σ -pointed category $(\mathbb{D}, \star_{\mathbb{D}})$ with pullbacks is said to satisfy condition P_n if for all X_1, \dots, X_n, Z , pointed cobase-change $(\alpha_Z)_! : \mathbb{D} \rightarrow \text{Pt}_Z(\mathbb{D})$ takes the object P_{X_1, \dots, X_n} to the object P_{X_1, \dots, X_n}^{+Z} .*

In particular, since $P_X = \star$ condition P_1 is void and just expresses that $(\alpha_Z)_!$ preserves the null-object. Since $P_{X,Y} = X \times Y$ condition P_2 expresses that $(\alpha_Z)_!$ preserves binary products. Therefore, the following result extends Corollary 4.4 ($n=1$) and Theorem 5.6 ($n=2$) to all positive integers.

Proposition 6.17. *The identity functor of a σ -pointed exact Mal'cev category \mathbb{D} is of degree $\leq n$ if and only if all objects are n -nilpotent and the Birkhoff subcategories $\text{Nil}^k(\mathbb{D})$ satisfy condition P_k for $1 \leq k \leq n$.*

Proof. Since the statement is true for $n=1$ by Corollary 4.4 we can argue by induction on n and assume that the statement is true up to level $n-1$. In particular, we can assume that $\text{Nil}^{n-1}(\mathbb{D})$ has an identity functor of degree $\leq n-1$. Let us then consider the following substitute of the main diagram 5.4:

$$\begin{array}{ccccc}
(X_1 + \cdots + X_n) \diamond Z & \xrightarrow{\theta_{X_1, \dots, X_n}^Z} & (X_1 + \cdots + X_n) + Z & \xrightarrow{\theta_{X_1, \dots, X_n}^Z} & (X_1 + \cdots + X_n) \times Z \\
\theta_{X_1, \dots, X_n}^Z \downarrow & & \downarrow \theta_{X_1, \dots, X_n}^Z & \textcircled{a} & \downarrow \theta_{X_1, \dots, X_n}^Z \\
P_{X_1, \dots, X_n} \diamond Z & \xrightarrow{\varphi_{X_1, \dots, X_n}^Z} & P_{X_1, \dots, X_n} + Z & \xrightarrow{\theta_{P_{X_1, \dots, X_n}}^Z} & P_{X_1, \dots, X_n} \times Z \\
\varphi_{X_1, \dots, X_n}^Z \downarrow & & \downarrow \phi_{X_1, \dots, X_n}^Z & \textcircled{b} & \downarrow \mu_{X_1, \dots, X_n}^Z \\
P_{X_1, \dots, X_n}^{\diamond Z} & \xrightarrow{\varphi_{X_1, \dots, X_n}^Z} & P_{X_1, \dots, X_n}^{+Z} & \xrightarrow{\theta_{P_{X_1, \dots, X_n}}^Z} & P_{X_1, \dots, X_n}^{\times Z}
\end{array}$$

in which the composite vertical morphisms from left to right are respectively

$$\theta_{X_1, \dots, X_n}^{\diamond Z} \text{ and } \theta_{X_1, \dots, X_n}^{+Z} \text{ and } \theta_{X_1, \dots, X_n}^{\times Z},$$

and the morphism μ_{X_1, \dots, X_n}^Z is the canonical isomorphism. Exactly as in the proof of Proposition 5.5 it follows that the identity functor of \mathbb{D} is of degree $\leq n$ if and only if squares (a) and (b) are pullback squares.

Square (b) is a pullback if and only if ϕ_{X_1, \dots, X_n}^Z is invertible which is the case precisely when condition P_n holds. By Proposition 3.12 and Theorem 6.15, square (a) is a pullback if and only if $\theta_{X_1, \dots, X_n}^Z$ is an affine (resp. central) extension, which is the case for all objects X_1, \dots, X_n precisely when \mathbb{D} is n -nilpotent. \square

Corollary 6.18. *A semi-abelian category has an identity functor of degree $\leq n$ if and only if either of the following three equivalent conditions is satisfied:*

- (a) *all objects are n -nilpotent, and the comparison maps*

$$\varphi_{X_1, \dots, X_n}^Z : P_{X_1, \dots, X_n} \diamond Z \rightarrow P_{X_1, \dots, X_n}^{\diamond Z}$$

are invertible for all objects X_1, \dots, X_n, Z ;

- (b) *the $(n+1)$ st cross-effects of the identity functor*

$$cr_{n+1}(X_1, \dots, X_n, Z) = K[\theta_{X_1, \dots, X_n, Z}]$$

vanish for all objects X_1, \dots, X_n, Z ;

- (c) *the co-smash product is of degree $\leq n-1$, i.e. the comparison maps*

$$\theta_{X_1, \dots, X_n}^{\diamond Z} : (X_1 + \cdots + X_n) \diamond Z \rightarrow P_{X_1, \dots, X_n}^{\diamond Z}$$

are invertible for all objects X_1, \dots, X_n, Z .

Proof. Condition (a) expresses that squares (a) and (b) of the main diagram are pullbacks. By Proposition 6.17 this amounts to an identity functor of degree $\leq n$.

For (b) note that by protomodularity the cross-effect $cr_{n+1}(X_1, \dots, X_n, Z)$ is trivial if and only if the regular epimorphism $\theta_{X_1, \dots, X_n, Z}$ is invertible.

The equivalence of conditions (b) and (c) follows from the isomorphism of kernels $K[\theta_{X_1, \dots, X_n, Z}] \cong K[\theta_{X_1, \dots, X_n}^{\diamond Z}]$. The latter is a consequence of the 3×3 -lemma which, applied to main diagram 6.17 and to square (n), yields a chain of isomorphisms:

$$K[\theta_{X_1, \dots, X_n}^{\diamond Z}] \cong K[K[\theta_{X_1, \dots, X_n}^{+Z}] \rightarrow K[\theta_{X_1, \dots, X_n}^{\times Z}]] \cong K[\theta_{X_1, \dots, X_n, Z}].$$

\square

6.19. Higher duality and multilinear cross-effects. –

In Section 5 we obtained a precise criterion for when 2-nilpotency implies quadraticity, namely algebraic distributivity, cf. Corollary 5.19. We now look for a similar criterion for when n -nilpotency implies an identity functor of degree $\leq n$. Proposition 6.17 gives us an explicit exactness condition in terms of certain limit-preservation properties (called P_n) of pointed cobase-change along initial maps. In order to exploit the latter we first need to dualize condition P_n into a colimit-preservation property, extending Proposition 5.18. Surprisingly, this dualization process yields the simple condition that in each variable, the functor $P_{X_1, \dots, -, \dots, X_n}$ takes binary sums to binary sums in the fibre over $P_{X_1, \dots, x, \dots, X_n}$. For n -nilpotent semi-abelian categories, this in turn amounts to the condition that the n -th cross-effect of the identity functor $cr_n(X_1, \dots, X_n) = K[\theta_{X_1, \dots, X_n}]$ is *multilinear*.

Such a characterization of degree n functors in terms of the multilinearity of their n -th cross-effect is already present in the original treatment of Eilenberg-Mac Lane [24] for functors between abelian categories. It plays also an important role in Goodwillie's [29] homotopical context (where however linearity has a slightly different meaning). The following lemma is known in contexts close to our's.

Lemma 6.20. *Let \mathbb{D} be a σ -pointed category and let \mathbb{E} be an abelian category. Any multilinear functor $F : \mathbb{D}^n \rightarrow \mathbb{E}$ has a diagonal $G : \mathbb{D} \xrightarrow{\Delta_n^n} \mathbb{D}^n \xrightarrow{F} \mathbb{E}$ of degree $\leq n$.*

Proof. This is a consequence of the decomposition formula of Eilenberg-Mac Lane for functors taking values in abelian categories, cf. Remark 6.2. Indeed, an induction on k shows that the k -th cross-effect of the diagonal $cr_k^G(X_1, \dots, X_k)$ is the direct sum of all terms $F(X_{j_1}, \dots, X_{j_n})$ such that the sequence (j_1, \dots, j_n) contains only integers $1, 2, \dots, k$, but each of them at least once. In particular,

$$cr_n^G(X_1, \dots, X_n) \cong \bigoplus_{\sigma \in \Sigma_n} F(X_{\sigma(1)}, \dots, X_{\sigma(n)})$$

and the cross-effects of G of order $> n$ vanish, whence G is of degree $\leq n$. \square

Lemma 6.21. *For each $n \geq 1$, the n -th cross-effect of the identity functor of a semi-abelian category preserves regular epimorphisms in each variable.*

Proof. The first cross-effect is the identity functor and the second cross-effect is the co-smash product. Proposition 1.19 and Lemma 1.14 imply that the co-smash product preserves regular epimorphisms in both variables.

The general case $n+1$ follows from the already treated case $n = 1$. By symmetry it suffices to establish the preservation property for the last variable which we shall denote Z . We have the following formula:

$$cr_{n+1}(X_1, \dots, X_n, Z) = cr_n^{\diamond Z}(X_1, \dots, X_n) \quad (n \geq 1)$$

where $cr_n^{\diamond Z}(X_1, \dots, X_n) = K[\theta_{X_1, \dots, X_n}^{\diamond Z}]$ denotes the n -th cross-effect of the functor $- \diamond Z$. Indeed, this kernel has already been identified with $K[\theta_{X_1, \dots, X_n, Z}]$ in the proofs of Corollaries 5.7 and 6.18. It is now straightforward to deduce preservation of regular epimorphisms in Z using that $(-) \diamond (-)$ preserves regular epimorphisms in both variables. \square

Corollary 6.22. *In a semi-abelian category, the image of a Higgins commutator $[X, \dots, X]$ of X under a regular epimorphism $f : X \twoheadrightarrow Y$ is the corresponding Higgins commutator $[Y, \dots, Y]$ of Y .*

Proof. The Higgins commutator of length n is the image of the diagonal n -th cross-effect $K[\theta_{X,\dots,X}]$ under the folding map $\delta_n^X : X + \dots + X \rightarrow X$, cf. Section 6.10. By Lemma 6.21, any regular epimorphism $f : X \twoheadrightarrow Y$ induces a regular epimorphism $K[\theta_{X,\dots,X}] \twoheadrightarrow K[\theta_{Y,\dots,Y}]$ on diagonal cross-effects, whence the result. \square

Note that the commutative square (n) of the beginning of this section induces the following pullback square

$$\begin{array}{ccc} P_{X_1,\dots,X_{n+1}} & \xrightarrow{\chi_{X_1,\dots,X_{n+1}}} & P_{X_1,\dots,X_{n+1}}^{+X_{n+1}} \\ \downarrow P_{X_1,\dots,X_n,\omega_{X_{n+1}}} & \downarrow P_{X_1,\dots,X_n,\alpha_{X_{n+1}}} & \downarrow P_{X_1,\dots,X_n}^{+\omega_{X_{n+1}}} \\ P_{X_1,\dots,X_n,\star} = X_1 + \dots + X_n & \xrightarrow{\theta_{X_1,\dots,X_n}} & P_{X_1,\dots,X_n}^{+\alpha_{X_{n+1}}} \end{array}$$

in which the identification $P_{X_1,\dots,X_{n-1},\star} = X_1 + \dots + X_{n-1}$ is exploited to give the left vertical morphisms names. Recall that $\alpha_X : \star \rightarrow X$ and $\omega_X : X \rightarrow \star$ denote the initial and terminal maps.

Proposition 6.23. *For any objects $X_1, \dots, X_{n-1}, Y, Z$ of a σ -pointed category $(\mathbb{D}, \star_{\mathbb{D}})$ with pullbacks consider the following diagram*

$$\begin{array}{ccc} P_{X_1,\dots,X_{n-1},Y} + Z & \xrightarrow{\rho_{X_1,\dots,X_{n-1},Y,Z}} & P_{X_1,\dots,X_{n-1},Y+Z} \\ \downarrow P_{X_1,\dots,X_{n-1},\omega_Y+Z} & & \downarrow P_{X_1,\dots,X_{n-1},\pi_Y^Z} \\ X_1 + \dots + X_{n-1} + Z & \xrightarrow{\theta_{X_1,\dots,X_{n-1},Z}} & P_{X_1,\dots,X_{n-1},Z} \end{array}$$

in which the horizontal map $\rho_{X_1,\dots,X_{n-1},Y,Z}$ is induced by the pair $P_{X_1,\dots,X_{n-1},\iota_Y^Z} : P_{X_1,\dots,X_{n-1},Y} \rightarrow P_{X_1,\dots,X_{n-1},Y+Z}$ and $P_{\alpha_{X_1},\dots,\alpha_{X_{n-1}},\iota_Y^Z} : Z \rightarrow P_{X_1,\dots,X_{n-1},Y+Z}$;

- (1) The functor $P_{X_1,\dots,X_{n-1},-} : \mathbb{D} \rightarrow \text{Pt}_{X_1+\dots+X_{n-1}}(\mathbb{D})$ preserves binary sums if and only if the upward-oriented square is a pushout for all objects Y, Z ;
- (2) the category \mathbb{D} satisfies condition P_n (cf. Definition 6.16) if and only if the downward-oriented square is a pullback for all objects $X_1, \dots, X_{n-1}, Y, Z$.

In particular, (1) and (2) hold simultaneously whenever $\theta_{X_1,\dots,X_{n-1},Z}$ is an affine extension for all objects X_1, \dots, X_{n-1}, Z .

Proof. The second assertion follows from the discussion in Section 3.2. For (1), observe that the left upward-oriented square of the following diagram

$$\begin{array}{ccccc} & & P_{X_1,\dots,X_{n-1},\iota_Y^Z} & & \\ & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \\ P_{X_1,\dots,X_{n-1},Y} & \xrightarrow{\quad} & P_{X_1,\dots,X_{n-1},Y} + Z & \xrightarrow{\rho_{X_1,\dots,X_{n-1},Y,Z}} & P_{X_1,\dots,X_{n-1},Y+Z} \\ \downarrow P_{X_1,\dots,X_{n-1},\alpha_Y} & & \downarrow & & \downarrow P_{X_1,\dots,X_{n-1},\pi_Y^Z} \\ X_1 + \dots + X_{n-1} & \xrightarrow{\quad} & X_1 + \dots + X_{n-1} + Z & \xrightarrow{\theta_{X_1,\dots,X_{n-1},Z}} & P_{X_1,\dots,X_{n-1},Z} \\ & & P_{X_1,\dots,X_{n-1},\alpha_Z} & & \end{array}$$

is a pushout so that the whole upward-oriented rectangle is a pushout if and only if the right upward-oriented square is a pushout, establishing (1).

For (2) observe that the right downward-oriented square of the following diagram

$$\begin{array}{ccccc}
 P_{X_1, \dots, X_{n-1}, Y} + Z & \xrightarrow{\rho_{X_1, \dots, X_{n-1}, Y, Z}} & P_{X_1, \dots, X_{n-1}, Y+Z} & \xrightarrow{\chi_{X_1, \dots, X_{n-1}, Y+Z} + Y+Z} & P_{X_1, \dots, X_{n-1}}^{+Y+Z} \\
 \updownarrow & & \updownarrow & & \updownarrow \\
 X_1 + \dots + X_{n-1} + Z & \xrightarrow{\theta_{X_1, \dots, X_{n-1}, Z}} & P_{X_1, \dots, X_{n-1}, Z} & \xrightarrow{\chi_{X_1, \dots, X_{n-1}, Z}} & P_{X_1, \dots, X_{n-1}}^{+Z} \\
 & \xrightarrow{\theta_{X_1, \dots, X_{n-1}}^{+Z}} & & &
 \end{array}$$

$P_{X_1, \dots, X_{n-1}}^{+\pi Y} \quad P_{X_1, \dots, X_{n-1}}^{+\iota Y}$

is a pullback (see below) so that the whole downward-oriented rectangle is a pullback if and only if the left downward-oriented square is a pullback. The whole downward-oriented rectangle is a pullback if and only if the comparison map

$$P_{X_1, \dots, X_{n-1}, Y} + Z \rightarrow P_{X_1, \dots, X_{n-1}, Y}^{+Z}$$

is invertible (i.e. if and only if condition P_n holds) since the following square is by definition a pullback in the fibre $\text{Pt}_Z(\mathbb{D})$:

$$\begin{array}{ccc}
 P_{X_1, \dots, X_{n-1}, Y}^{+Z} & \xrightarrow{\chi_{X_1, \dots, X_{n-1}, Y}^{+Z}} & P_{X_1, \dots, X_{n-1}}^{+Y+Z} \\
 \updownarrow P_{X_1, \dots, X_{n-1}, \pi Y}^{+Z} & & \updownarrow P_{X_1, \dots, X_{n-1}}^{+\pi Y} \\
 X_1 + \dots + X_{n-1} + Z & \xrightarrow{\theta_{X_1, \dots, X_{n-1}}^{+Z}} & P_{X_1, \dots, X_{n-1}}^{+Z}
 \end{array}$$

Thus (2) is established. The pullback property of the right square above follows finally from considering the following diagram

$$\begin{array}{ccc}
 P_{X_1, \dots, X_{n-1}, Y+Z} & \xrightarrow{\chi_{X_1, \dots, X_{n-1}, Y+Z} + Y+Z} & P_{X_1, \dots, X_{n-1}}^{+Y+Z} \\
 \updownarrow & & \updownarrow \\
 P_{X_1, \dots, X_{n-1}, Z} & \xrightarrow{\chi_{X_1, \dots, X_{n-1}, Z}} & P_{X_1, \dots, X_{n-1}}^{+Z} \\
 \updownarrow & & \updownarrow \\
 X_1 + \dots + X_{n-1} & \xrightarrow{\theta_{X_1, \dots, X_{n-1}}^{+Z}} & P_{X_1, \dots, X_{n-1}}
 \end{array}$$

in which whole rectangle and lower square are downward-oriented pullbacks. \square

Theorem 6.24. *Let \mathbb{D} be an n -nilpotent σ -pointed exact Mal'cev category such that the identity functor of $\text{Nil}^{n-1}(\mathbb{D})$ is of degree $\leq n-1$.*

The following properties are equivalent:

- (a) *the identity functor of \mathbb{D} is of degree $\leq n$;*
- (b) *the category \mathbb{D} satisfies condition P_n (cf. Definition 6.16);*
- (c) *the functor $P_{X_1, \dots, X_{n-1}, -} : \mathbb{D} \rightarrow \text{Pt}_{X_1 + \dots + X_{n-1}}(\mathbb{D})$ preserves binary sums for all objects X_1, \dots, X_{n-1} .*

If \mathbb{D} is semi-abelian then the former properties are also equivalent to:

- (d) *the n -th cross-effect of the identity is coherent in each variable;*
- (e) *the n -th cross-effect of the identity is linear in each variable;*
- (f) *the diagonal n -th cross-effect of the identity is a functor of degree $\leq n$.*

Proof. It follows from Proposition 6.17 that properties (a) and (b) are equivalent, while properties (b) and (c) are equivalent by Theorem 6.15 and Proposition 6.23. For the equivalence between (c) and (d), note first that the n -th cross-effect preserves regular epimorphisms in each variable by Lemma 6.21 so that coherence (in the last variable) amounts to the property that the canonical map

$$cr_n(X_1, \dots, X_{n-1}, Y) + cr_n(X_1, \dots, X_{n-1}, Z) \rightarrow cr_n(X_1, \dots, X_{n-1}, Y + Z)$$

is a regular epimorphism. Since by Theorem 6.15 for $W = Y, Z, Y + Z$ the regular epimorphism $X_1 + \dots + X_{n-1} + W \twoheadrightarrow P_{X_1, \dots, X_{n-1}, W}$ is an affine extension, Proposition 3.15 establishes the equivalence between (c) and (d). Finally, consider the following commutative diagram in $\text{Nil}^1(\mathbb{D}) = \text{Ab}(\mathbb{D})$

$$\begin{array}{ccc} I^1(cr_n(X_{1\dots n-1}, Y) + cr_n(X_{1\dots n-1}, Z)) & \xrightarrow{\cong} & cr_n(X_{1\dots n-1}, Y) \times cr_n(X_{1\dots n-1}, Z) \\ \downarrow & & \uparrow \\ cr_n(X_{1\dots n-1}, Y + Z) & \xrightarrow{cr_n(X_1, \dots, X_{n-1}, \theta_{Y,Z})} & cr_n(X_{1\dots n-1}, Y \times Z) \end{array}$$

in which the upper horizontal map is invertible because the n -th cross-effect takes values in abelian group objects. It follows that the left vertical map is a section so that property (d) is equivalent to the invertibility of this left vertical map. Therefore, (d) is equivalent to the invertibility of the diagonal map

$$cr_n(X_1, \dots, X_{n-1}, Y + Z) \rightarrow cr_n(X_1, \dots, X_{n-1}, Y) \times cr_n(X_1, \dots, X_{n-1}, Z)$$

which expresses linearity in the last variable, i.e. property (e).

Property (e) implies property (f) by Lemma 6.20. It suffices now to prove that (f) implies (a). The Higgins commutator $[X, \dots, X]$ of length n is the image of diagonal n -th cross-effect $cr_n(X, \dots, X)$ under the n -th folding map $\delta_n^X : X + \dots + X \rightarrow X$. The Higgins commutator of length n is thus a quotient-functor of the diagonal n -th cross-effect and as such a functor of degree $\leq n$ by Theorem 6.26a. Corollary 6.13 and Remark 2.15 imply that the kernel $K[\eta_X^{n-1} : X \twoheadrightarrow I^{n-1}(X)]$ (considered as a functor in X) is a subfunctor of the Higgins commutator of length n and hence, again by Theorem 6.26a, a functor of degree $\leq n$. It follows then from the short exact sequence of endofunctors

$$\star \longrightarrow K[\eta^{n-1}] \longrightarrow id_{\mathbb{D}} \longrightarrow I^{n-1} \longrightarrow \star$$

(by a third application of Theorem 6.26a) that the identity functor of \mathbb{D} is also of degree $\leq n$, whence (f) implies (a). \square

6.25. Homogeneous nilpotency towers. –

One of the starting points of this article was the existence of a functorial nilpotency tower for any σ -pointed exact Mal'cev category \mathbb{E} , cf. Section 2.16. It is not surprising that for a semi-abelian category \mathbb{E} the successive kernels of the nilpotency tower capture the essence of the whole tower.

To make this more precise, we denote by

$$L_{\mathbb{E}}(X) = \bigoplus_{n \geq 1} L_{\mathbb{E}}^n(X) = \bigoplus_{n \geq 1} K[I^n(X) \twoheadrightarrow I^{n-1}(X)] \in \text{Ab}(\mathbb{E})$$

the graded abelian group object defined by the successive kernels. This construction is a functor in X . The *nilpotency tower* of \mathbb{E} is said to be *homogeneous* if for each n , the n -th kernel functor $L_{\mathbb{E}}^n : \mathbb{E} \rightarrow \text{Ab}(\mathbb{E})$ is a functor of degree $\leq n$.

The degree of a functor does not change under composition with conservative left exact functors. We can therefore consider $L_{\mathbb{E}}^n$ as an endofunctor of \mathbb{E} . Observe also that the binary sum in $\text{Nil}^n(\mathbb{E})$ is obtained as the reflection of the binary sum in \mathbb{E} . This implies that the degree of $L_{\mathbb{E}}^n$ is the same as the degree of $L_{\text{Nil}^n(\mathbb{E})}^n$. We get the following short exact sequence of endofunctors of $\text{Nil}^n(\mathbb{E})$

$$\star \longrightarrow L_{\text{Nil}^n(\mathbb{E})}^n \longrightarrow \text{id}_{\text{Nil}^n(\mathbb{E})} \longrightarrow I_{\mathbb{E}}^{n,n-1} \longrightarrow \star$$

where the last term is the relative Birkhoff reflection $I_{\mathbb{E}}^{n,n-1} : \text{Nil}^n(\mathbb{E}) \rightarrow \text{Nil}^{n-1}(\mathbb{E})$.

A more familiar way to express the successive kernels $L_{\mathbb{E}}^n(X)$ of the nilpotency tower of X is to realize them as subquotients of the lower central series of X . Indeed, the 3×3 -lemma implies that there is a short exact sequence

$$\star \longrightarrow L_{\mathbb{E}}^n(X) = \gamma_n(X)/\gamma_{n+1}(X) \longrightarrow X/\gamma_{n+1}(X) \longrightarrow X/\gamma_n(X) \longrightarrow \star$$

where $\gamma_{n+1}(X)$ denotes the iterated Huq commutator of X of length $n+1$, i.e. the kernel of the n -th Birkhoff reflection $\eta_X^n : X \rightarrow I^n(X)$, cf. Remark 2.15.

The conclusion of the following theorem is folklore among those who are familiar with Goodwillie calculus in homotopy theory (cf. [4, 29]). Ideally, we would have liked to establish Theorem 6.26c by checking inductively one of the conditions of Theorem 6.24 without using any computation involving elements.

Theorem 6.26. *Let \mathbb{E} be a semi-abelian category.*

- (a) *For any short exact sequence $\star \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow \star$ of endofunctors of \mathbb{E} , F is of degree $\leq n$ if and only if F_1 and F_2 are both of degree $\leq n$;*
- (b) *The nilpotency tower of \mathbb{E} is homogeneous if and only if the identity functors of $\text{Nil}^n(\mathbb{E})$ are of degree $\leq n$ for all n ;*
- (c) *The category of groups and the category of Lie algebras have homogeneous nilpotency towers.*

Proof. For (a) we need the following *cogluing lemma* for regular epimorphisms in regular categories: for any quotient-map of cospans

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ f \downarrow & & h \downarrow & & \downarrow g \\ X' & \longrightarrow & Z' & \longleftarrow & Y' \end{array}$$

in which the left naturality square is a *regular pushout* (cf. Section 1.8), the induced map on pullbacks $f \times_h g : X \times_Z Y \rightarrow X' \times_{Z'} Y'$ is again a regular epimorphism. Indeed, a diagram chase shows that the following square

$$\begin{array}{ccc} X & \longleftarrow & X \times_Z Y \\ \downarrow & & \downarrow \text{dotted} \\ X' \times_{Z'} Z & \longleftarrow & (X' \times_{Z'} Y') \times_{Y'} Y \end{array}$$

is a pullback. The left vertical map is a regular epimorphism by assumption so that the right vertical map is a regular epimorphism as well. Since g is a regular epimorphism, the projection $(X' \times_{Z'} Y') \times_{Y'} Y \rightarrow X' \times_{Z'} Y'$ is again a regular epimorphism so that $f \times_h g$ is the composite of two regular epimorphisms.

The limit construction $P_{X_1, \dots, X_{n+1}}^F$ is an iterated pullback along *split* epimorphisms. Therefore, Corollary 1.10 and the cogluing lemma show inductively that the morphism $P_{X_1, \dots, X_{n+1}}^F \rightarrow P_{X_1, \dots, X_{n+1}}^{F_2}$ induced by the quotient-map $F \twoheadrightarrow F_2$ is a regular epimorphism. The 3×3 -lemma yields then the exact 3×3 -square

$$\begin{array}{ccccccc}
 & & \star & & \star & & \star \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \star & \longrightarrow & cr_{n+1}^{F_1}(X_1, \dots, X_{n+1}) & \longrightarrow & cr_{n+1}^F(X_1, \dots, X_{n+1}) & \longrightarrow & cr_{n+1}^{F_2}(X_1, \dots, X_{n+1}) \longrightarrow \star \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \star & \longrightarrow & F_1(X_1 + \dots + X_{n+1}) & \longrightarrow & F(X_1 + \dots + X_{n+1}) & \longrightarrow & F_2(X_1 + \dots + X_{n+1}) \longrightarrow \star \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \star & \longrightarrow & P_{X_1, \dots, X_{n+1}}^{F_1} & \longrightarrow & P_{X_1, \dots, X_{n+1}}^F & \longrightarrow & P_{X_1, \dots, X_{n+1}}^{F_2} \longrightarrow \star \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \star & & \star & & \star
 \end{array}$$

from which (a) immediately follows, cf. Corollary 6.18b.

For (b) we can assume inductively that $\text{Nil}^{n-1}(\mathbb{E})$ has an identity functor of degree $\leq n-1$ so that the Birkhoff reflection $I_{\mathbb{E}}^{n, n-1} : \text{Nil}^n(\mathbb{E}) \rightarrow \text{Nil}^{n-1}(\mathbb{E})$ is of degree $\leq n-1$, and finally $I_{\mathbb{E}}^{n, n-1}$ is also of degree $\leq n-1$ when considered as an endofunctor of $\text{Nil}^n(\mathbb{E})$. Statement (b) follows then from (a) by induction on n .

For (c) we treat the group case, the Lie algebra case being very similar. In the category of groups, the graded object $L_{\text{Grp}}(X)$ is a *graded Lie ring* with Lie bracket $[-, -] : L_{\text{Grp}}^m(X) \otimes L_{\text{Grp}}^n(X) \rightarrow L_{\text{Grp}}^{m+n}(X)$ induced by the *commutator map* $(x, y) \mapsto xyx^{-1}y^{-1}$ in X . This graded Lie ring is generated by its elements of degree 1, cf. Lazard [51, Section I.2]. In particular, there is a regular epimorphism of abelian groups $L_{\text{Grp}}^1(X)^{\otimes n} \rightarrow L_{\text{Grp}}^n(X)$ which is natural in X . The functor which assigns to X the tensor power $L_{\text{Grp}}^1(X)^{\otimes n}$ is the diagonal of a multilinear abelian-group-valued functor in n variables, and hence a functor of degree $\leq n$ by Lemma 6.20. It follows from (a) that its quotient-functor L_{Grp}^n is of degree $\leq n$ as well, whence the homogeneity of the nilpotency tower in the category of groups. \square

Theorem 6.27. *Let \mathbb{E} be a semi-abelian category.*

The following conditions are equivalent:

- (a) *The nilpotency tower of \mathbb{E} is homogeneous;*
- (b) *For each n , the n -th Birkhoff reflection $I^n : \mathbb{E} \rightarrow \text{Nil}^n(\mathbb{E})$ is of degree $\leq n$;*
- (c) *For each n , an object of \mathbb{E} is n -nilpotent if and only if it is n -folded;*
- (d) *For each object X of \mathbb{E} , iterated Huq commutator $[X, [X, \dots, X] \dots]$ and Higgins commutator $[X, X, \dots, X]$ of same length coincide.*

If \mathbb{E} satisfies one and hence all of these conditions, then so does any reflective Birkhoff subcategory of \mathbb{E} . If \mathbb{E} is algebraically extensive and satisfies one and hence all of these conditions then so does the fibre $\text{Pt}_X(\mathbb{E})$ over any object X of \mathbb{E} .

Proof. We have already seen that (b) is equivalent to condition (b) of Theorem 6.26, which implies the equivalence between (a) and (b). Propositions 6.5 and 6.14 show that (b) implies (c), while Theorem 6.8 shows that (c) implies (b). The equivalence between (c) and (d) is proved in exactly the same way as Corollary 5.22.

Let \mathbb{D} be a reflective Birkhoff subcategory of \mathbb{E} . We shall show that \mathbb{D} inherits (c) from \mathbb{E} . By Proposition 6.14, it suffices to show that in \mathbb{D} each n -nilpotent object X is n -folded. Since the inclusion $\mathbb{D} \hookrightarrow \mathbb{E}$ is left exact, it preserves n -nilpotent objects so that X is n -nilpotent in \mathbb{E} , and hence by assumption n -folded in \mathbb{E} . The Birkhoff reflection $\mathbb{E} \rightarrow \mathbb{D}$ preserves sums and the limit construction $P_{X_1, \dots, X_{n+1}}$ by an iterated application of Proposition 2.10. Therefore, X is indeed n -folded in \mathbb{D} .

By Lemma 5.12 algebraic extensivity implies that all pointed base-change functors are exact. By Lemma 2.17 this implies that the following square of functors

$$\begin{array}{ccc} \mathrm{Pt}_X(\mathbb{E}) & \xrightarrow{I_{\mathrm{Pt}_X(\mathbb{E})}^n} & \mathrm{Nil}^n(\mathrm{Pt}_X(\mathbb{E})) \\ \omega_X^* \downarrow & & \downarrow \omega_X^* \\ \mathbb{E} & \xrightarrow{I_{\mathbb{E}}^n} & \mathrm{Nil}^n(\mathbb{E}) \end{array}$$

commutes up to isomorphism. The vertical functors are exact and conservative. Therefore, if $I_{\mathbb{E}}^n$ is of degree $\leq n$ then $I_{\mathrm{Pt}_X(\mathbb{E})}^n$ is of degree $\leq n$ as well. \square

6.28. On Moufang loops and triality groups. –

We end this article by giving an example of a semi-abelian category in which 2-foldedness is not equivalent to 2-nilpotency, namely the semi-abelian variety of Moufang loops. In particular, the semi-abelian subvariety of 2-nilpotent Moufang loops is neither quadratic (cf. Proposition 6.5) nor algebraically distributive (cf. Corollaries 5.19 and 5.22). The nilpotency tower of the semi-abelian category of Moufang loops is thus inhomogeneous (cf. Theorem 6.27). Nevertheless, the category of Moufang loops fully embeds into the category of *triality groups* [23, 28, 37] which, as we will see, is a semi-abelian category with homogeneous nilpotency tower.

Recall [16] that a *loop* is a unital magma $(L, \cdot, 1)$ such that left and right translation by any element $z \in L$ are bijective. A *Moufang loop* [55] is a loop L such that $(z(xy))z = (zx)(yz) = z((xy)z)$ for all $x, y, z \in L$. Moufang loops form a semi-abelian variety which contains the variety of groups as a reflective Birkhoff subvariety. Moufang loops share many properties of groups, but the lack of a full associative law complicates the situation. The main example of a non-associative Moufang loop is the set of invertible elements of a non-associative *alternative* algebra (i.e. in characteristic $\neq 2$ a unital algebra in which the difference $(xy)z - x(yz)$ alternates in sign whenever two variables are permuted). In particular, the set \mathbb{O}^* of non-zero *octonions* forms a Moufang loop. Taking the standard real basis of the octonions together with their additive inverses yields a Moufang subloop

$$\mathbb{O}_{16} = \{\pm 1, \pm e_1, \dots, \pm e_7\}$$

with sixteen elements. We will see that \mathbb{O}_{16} is 2-nilpotent, but not 2-folded.

By Moufang's theorem [55], any Moufang loop, which can be generated by two elements, is associative and hence a group. In particular, for any element of a Moufang loop, left and right inverse coincide. The kernel of the reflection of a Moufang loop L into the category of groups is the so-called *associator subloop* $[L, L, L]_{ass}$ of L . For a Moufang loop L , the associator subloop is generated by the elements of the form $[x, y, z] = ((xy)z)(x(yz))^{-1}$. Such an “associator” satisfies $[1, y, z] = [x, 1, z] = [x, y, 1] = 1$ and is thus *3-reducible*, cf. Remark 6.4. This implies that for a Moufang loop L , the associator subloop $[L, L, L]_{ass}$ is contained

in the ternary Higgins commutator $[L, L, L]$, cf. Proposition 6.1 and Section 6.10. In conclusion, *any 2-folded Moufang loop has a trivial associator subloop and is therefore a 2-folded group*. In particular, \mathbb{O}_{16} cannot be 2-folded since \mathbb{O}_{16} is not a group. One can actually show that $[\mathbb{O}_{16}, \mathbb{O}_{16}, \mathbb{O}_{16}] = \{\pm 1\}$. On the other hand, the centre of \mathbb{O}_{16} is also $\{\pm 1\}$, and the quotient by the centre $\mathbb{O}_{16}/\{\pm 1\}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. This implies that \mathbb{O}_{16} is 2-nilpotent, i.e. $[\mathbb{O}_{16}, [\mathbb{O}_{16}, \mathbb{O}_{16}]] = \{1\}$.

The variety of Moufang loops is interesting with respect to the existence of centralizers. Since algebraic distributivity fails, such centralizers do not exist for general subloops, cf. [14]. Nevertheless, each Moufang loop L has a *centre* $Z(L)$ in the sense of Section 1.21, i.e. a centralizer $Z(1_L)$ for its identity $1_L : L \rightarrow L$. This centre $Z(L)$ is a *normal* subloop of L , and is the intersection $Z(L) = M(L) \cap N(L)$ of the *Moufang centre* $M(L) = \{z \in L \mid zx = xz \forall x \in L\}$ with the so-called *nucleus* $N(L) = \{z \in L \mid [z, x, y] = [x, z, y] = [x, y, z] = 1 \forall x, y \in L\}$, cf. Bruck [16].

Groups with triality have been introduced in the context of Moufang loops by Glauberman [28] and Doro [23]. A *triality* on a group G_0 is an action (by automorphisms) of the symmetric group \mathfrak{S}_3 on three letters such that for all $g \in G_0$ and $\sigma \in \mathfrak{S}_3$ (resp. $\rho \in \mathfrak{S}_3$) of order 2 (resp. 3), the identity

$$[\sigma, g](\rho.[\sigma, g])(\rho^2.[\sigma, g]) = 1$$

holds where $[\sigma, g] = (\sigma.g)g^{-1}$. We denote the split epimorphism associated to the group action by $p : G_0 \rtimes \mathfrak{S}_3 \rightrightarrows \mathfrak{S}_3 : i$ and call it the associated *triality group*. The defining relations for a group with triality are equivalent to the following condition on the associated triality group $p : G \rightrightarrows \mathfrak{S}_3 : i$ (cf. Liebeck [52] and Hall [37]):

for any two special elements $g, h \in G$ such that $p(g) \neq p(h)$ one has $(gh)^3 = 1$

where $g \in G$ is called *special* if g is conjugate in G to some element of order 2 in $i(\mathfrak{S}_3)$. For the obvious notion of morphism, the category TriGrp_\star of triality groups is a full subcategory of the fibre $\text{Pt}_{\mathfrak{S}_3}(\text{Grp})$ over the symmetric group \mathfrak{S}_3 .

The category TriGrp_\star is closed under taking subobjects, products and quotients in $\text{Pt}_{\mathfrak{S}_3}(\text{Grp})$. Moreover, quotienting out the normal subgroup generated by the products $(gh)^3$ for all pairs of special elements (g, h) such that $p(g) \neq p(h)$ defines a reflection $\text{Pt}_{\mathfrak{S}_3}(\text{Grp}) \rightarrow \text{TriGrp}_\star$. Therefore, TriGrp_\star is a reflective Birkhoff category of $\text{Pt}_{\mathfrak{S}_3}(\text{Grp})$. Since the category of groups is an algebraically extensive semi-abelian category (cf. Section 5.8) with homogeneous nilpotency tower (cf. Theorem 6.26), so is its fibre $\text{Pt}_{\mathfrak{S}_3}(\text{Grp})$ by Lemma 5.15 and Theorem 6.27. The reflective Birkhoff subcategory TriGrp_\star formed by the triality groups is thus also a semi-abelian category with homogeneous nilpotency tower, again by Theorem 6.27.

This result is remarkable because the category of triality groups contains the category of Moufang loops as a full *coreflective* subcategory, and the latter has an inhomogeneous nilpotency tower. The embedding of Moufang loops and its right adjoint have been described by Doro [23] for groups with triality, and by Hall [37] for the associated triality groups, see also Grishkov-Zavarnitsine [34]. Moufang loops can thus up to equivalence of categories be identified with triality groups for which the counit of the adjunction is invertible. Considering them inside the category of triality groups permits the construction of a homogeneous nilpotency tower.

REFERENCES

- [1] H.-J. Baues and T. Pirashvili – *Quadratic endofunctors of the category of groups*, Adv. Math. **141** (1999), 167–206. [32](#)

- [2] C. Berger – *Algebraic and homotopical nilpotency*, talk at CT2015, available at <http://math.unice.fr/~cberger/CT2015.pdf>. **5**
- [3] I. Bernstein and T. Ganea – *Homotopical nilpotency*, Illinois J. Math. **5** (1961), 99–130. **5**
- [4] G. Biedermann and B. Dwyer – *Homotopy nilpotent groups*, Algebr. Geom. Topol. **10** (2010), 33–61. **5, 56**
- [5] F. Borceux and D. Bourn – *Mal'cev, protomodular, homological and semi-abelian categories*, Math. Appl. **566**, Kluwer Acad. Publ., 2004. **3, 5, 6, 11, 12, 22, 25, 26, 27, 29, 32, 38, 42**
- [6] D. Bourn – *Normalization equivalence, kernel equivalence and affine categories*, Lect. Notes Math. **1488**, Springer Verlag 1991, 43–62. **1, 2, 12, 21**
- [7] D. Bourn – *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Struct. **4** (1996), 307–327. **1, 2, 5, 9, 11, 13, 21, 22, 27**
- [8] D. Bourn – *The denormalized 3×3 lemma*, J. Pure Appl. Algebra **177** (2003), 113–129. **1, 8**
- [9] D. Bourn – *Commutator theory in regular Mal'cev categories*, AMS Fields Inst. Commun. **43** (2004) 61–75. **5, 6, 7**
- [10] D. Bourn – *Commutator theory in strongly protomodular categories*, Theory Appl. Categ. **13** (2004), 27–40. **12**
- [11] D. Bourn – *On the monad of internal groupoids*, Theory Appl. Categ. **28** (2013), 150–165. **37**
- [12] D. Bourn and M. Gran – *Central extensions in semi-abelian categories*, J. Pure Appl. Algebra **175** (2002), 31–44. **5, 6**
- [13] D. Bourn and M. Gran – *Centrality and connectors in Maltsev categories*, Algebra Universalis **48** (2002), 309–331. **5, 6**
- [14] D. Bourn and J.R.A. Gray – *Aspects of algebraic exponentiation*, Bull. Belg. Math. Soc. **19** (2012), 823–846. **3, 12, 13, 36, 59**
- [15] D. Bourn and D. Rodelo – *Comprehensive factorization and I-central extensions*, J. Pure Appl. Algebra **216** (2012), 598–617. **18, 21**
- [16] R. H. Bruck – *A survey of binary systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete **20**, Springer Verlag 1958. **2, 58, 59**
- [17] A. Carboni and G. Janelidze – *Smash product of pointed objects in left extensive categories*, J. Pure Appl. Algebra **183** (2003), 27–43. **2, 3, 27, 33, 41**
- [18] A. Carboni, G. M. Kelly and M. C. Pedicchio – *Some remarks on Mal'tsev and Goursat categories*, Appl. Categ. Struct. **1** (1993), 385–421. **1, 4, 5, 9, 13, 28**
- [19] A. Carboni, S. Lack and R. F. C. Walters – *Introduction to extensive and distributive categories*, J. Pure Appl. Algebra **84** (1993), 145–158. **36**
- [20] A. Carboni, J. Lambek, M. C. Pedicchio, – *Diagram chasing in Malcev categories*, J. Pure Appl. Algebra **69** (1991), 271–284. **1**
- [21] A. Cigoli, J. R. A. Gray, T. Van der Linden – *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), 1864–1905. **3, 4, 36, 37, 40**
- [22] C. Costoya, J. Scherer, A. Viruel – *A torus theorem for homotopy nilpotent groups*, arXiv:1504.06100. **5**
- [23] S. Doro – *Simple Moufang loops*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 377–392. **4, 58, 59**
- [24] S. Eilenberg and S. Mac Lane – *On the groups $H(\pi, n)$. II. Methods of computation*, Ann. of Math. (2) **60** (1954), 49–139. **3, 32, 41, 52**
- [25] R. Eldred – *Goodwillie calculus via adjunction and LS cocategory*, arXiv:1209.2384. **5**
- [26] T. Everaert and T. Van der Linden – *Baer invariants in semi-abelian categories I: general theory*, Theory Appl. Categ. **12** (2004), 1–33. **19**
- [27] R. S. Freese and R. N. McKenzie – *Commutator theory for congruence modular varieties*, London Math. Soc. Lect. Note Series **125**, Cambridge Univ. Press, Cambridge, 1987. **1**
- [28] G. Glauberman – *On loops of odd order II*, J. Algebra **8** (1968), 383–414. **4, 58, 59**
- [29] T. G. Goodwillie – *Calculus III. Taylor series*, Geom. Topol. **7** (2003), 645–711. **1, 2, 32, 33, 41, 52, 56**
- [30] M. Gran – *Central extensions and internal groupoids in Maltsev categories*, J. Pure Appl. Alg. **155** (2001), 139–166. **13**
- [31] M. Gran, G. Kadjo and J. Vercruysse – *A torsion theory in the category of cocommutative Hopf algebras*, arXiv:1502.03130. **2**
- [32] M. Gran and D. Rodelo – *Beck-Chevalley condition and Goursat categories*, arXiv:1512.04066. **5**

- [33] M. Gran and T. Van der Linden – *On the second cohomology group in semi-abelian categories*, J. Pure Appl. Algebra **212** (2008), 636–651. [12](#), [49](#)
- [34] A. N. Grishkov and A. V. Zavarnitsine – *Groups with triality*, J. Algebra Appl. **5** (2006), 441–463. [59](#)
- [35] J. R. A. Gray – *Algebraic exponentiation in general categories*, Appl. Categ. Struct. **20** (2012), 543–567. [36](#)
- [36] J. R. A. Gray – *Algebraic exponentiation for categories of Lie algebras*, J. Pure Appl. Algebra **216** (2012), 1964–1967. [36](#)
- [37] J. I. Hall – *Central automorphisms, Z^* -theorems, and loop structures*, Quasigroups and Related Systems **19** (2011), 69–108. [4](#), [58](#), [59](#)
- [38] M. Hartl and B. Loiseau – *On actions and strict actions in homological categories*, Theory Appl. Categ. **27** (2013), 347–392. [1](#), [3](#), [27](#), [32](#), [33](#), [40](#), [41](#), [47](#)
- [39] M. Hartl and T. Van der Linden – *The ternary commutator obstruction for internal crossed modules*, Adv. Math. **232** (2013), 571–607. [1](#), [3](#), [27](#), [32](#), [33](#), [40](#), [41](#), [47](#)
- [40] M. Hartl and C. Vespa – *Quadratic functors on pointed categories*, Adv. Math. **226** (2011), 3927–4010. [32](#)
- [41] P. J. Higgins – *Groups with multiple operators*, Proc. London Math. Soc. **6** (1956), 366–416. [1](#), [47](#)
- [42] M. Hovey – *Lusternik-Schnirelmann cocategory*, Illinois J. Math. **37** (1993), 224–239. [5](#), [41](#)
- [43] S. A. Huq – *Commutator, nilpotency and solvability in categories*, Quart. J. Math. Oxford **19** (1968), 363–389. [1](#), [12](#), [19](#)
- [44] G. Janelidze and G. M. Kelly – *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra **97** (1994), 135–161. [5](#), [16](#)
- [45] G. Janelidze and G. M. Kelly – *Central extensions in universal algebra: a unification of three notions*, Algebra Universalis **44** (2000), 123–128. [5](#)
- [46] G. Janelidze and G. M. Kelly – *Central extensions in Mal'tsev varieties*, Theory Appl. Categ. **7** (2000), 219–226. [5](#)
- [47] G. Janelidze, L. Márki and W. Tholen – *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), 367–386. [1](#)
- [48] G. Janelidze, M. Sobral and W. Tholen – *Beyond Barr exactness: effective descent morphisms*, Cambridge Univ. Press, Encycl. Math. Appl. **97** (2004), 359–405. [21](#)
- [49] B. Johnson and R. McCarthy – *A classification of degree n functors I/II* , Cah. Topol. Géom. Différ. Catég. **44** (2003), 2–38, 163–216. [32](#)
- [50] S. Lack – *The 3-by-3 lemma for regular Goursat categories*, Homology, Homotopy Appl. **6** (2004), 1–3. [1](#)
- [51] M. Lazard – *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. E. N. S. **71** (1954), 101–190. [4](#), [57](#)
- [52] M. W. Liebeck – *The classification of finite simple Moufang loops*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 33–47. [59](#)
- [53] A. I. Mal'cev – *On the general theory of algebraic systems*, Mat. Sbornik N. S. **35** (1954), 3–20. [1](#)
- [54] S. Mantovani and G. Metere – *Normalities and commutators*, J. of Algebra **324** (2010), 2568–2588. [1](#), [28](#), [47](#)
- [55] R. Moufang – *Zur Struktur von Alternativkörpern*, Math. Ann. **110** (1935), 416–430. [58](#)
- [56] M. C. Pedicchio – *A categorical approach to commutator theory*, J. of Algebra **177** (1995), 647–657. [6](#)
- [57] J. Penon – *Sur les quasi-topos*, Cah. Topol. Géom. Diff. **18** (1977), 181–218. [21](#)
- [58] D. Quillen – *Homotopical algebra*, Lect. Notes Math. **43**, Springer Verlag 1967. [4](#)
- [59] J. D. H. Smith – *Mal'cev varieties*, Lect. Notes Math. **554**, Springer Verlag 1976. [1](#), [6](#)

UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS, LAB. J. A. DIEUDONNÉ, UMR N° 7351 DU CNRS, PARC VALROSE, 06108 NICE, FRANCE. *E-mail:* cberger@math.unice.fr

UNIVERSITÉ DU LITTORAL, LAB. DE MATHÉMATIQUES, 50 RUE F. BUISSON, BP 699, 62228 CALAIS CEDEX, FRANCE. *E-mail:* Dominique.Bourn@lmpa.univ-littoral.fr